Chaotic Fractals with Multivalued Logic in Cellular Automata

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Abstract: This report deals with the application of multi-valued logic in cellular automata. A four valued logic system with dibit representation has been considered in this case. The general properties and their relations to build such logical systems are also investigated and the question of implementation of this logical system in cellular automata environment has also been studied. It is shown, that chaotic fractals i.e, fractals as function of initial conditions are formed in such cases. It is interesting to note also that fractals so formed are multifractals and thus may have applications in analyzing natural fractal formation.

Subject. terms: Multivalued Logic, Cellular Automata, Fractals.

1. Introduction

Two valued logic includes those logical systems, which are based on the hypothesis of two valued propositions. This two valued conception of logic is expressed, for instance, by constructing the propositional calculus in such a way that all tautologies of the two-valued algebra of propositions are derivable in it, so that it is deductively complete with respect to the two-valued algebra of proposition. By many –valued logical systems we mean many-valued constructions in the logic of proposition and predicates. It includes along with construction of such logical systems, the investigation of their properties and relations [1]. Historically, the first many-valued propositional logic is the system constructed by Lukasiewicz[2].

Starting with the analysis of modal propositions, Lukasiewicz came to the conclusion that two-valued logic is insufficient for the description of the mutual relations and properties of these propositions that we need here a logic in which, besides the classical truth values 'true' and 'false', there is a third value 'possible', 'neutral'(a neutral intermediate value). Later Lukasiewicz [3] revised his point of view on modal logic and applied instead of a three valued logic, a four valued one in which the laws of two-valued proposition logic remain valid. In the present report we have has shown its importance in application to cellular automata.

Cellular automata are mathematical idealization of physical systems in which space and time are discrete, and physical quantities take on finite set of discrete values. A cellular automation consists of a regular uniform lattice (or 'array'), usually infinite in extent, with a discrete variable at each site (cell). The state of a cellular automation is completely specified by the variables at each site. A cellular automation evolves in discrete time steps, with the value of the variable at one site being affected by the variables at sites in its 'neighborhood' on the previous time step[4]. The 'neighborhood' of a site is typically taken to be the site itself and all immediately adjacent sites.

The variables at each site update synchronously based on the values of the variables in their neighborhood at the preceding time step and according to a definite set of 'local rules'. The development of structure and pattern in biological systems often appears to be governed by very simple local rules and thus may be described by cellular automation model[5]. Any physical system satisfying differential equation may be approximated as a cellular automation by introducing fine differences and discrete variables. Multivalued logic may play an important role in solving such differential equations. In the present report the different possibilities of such applications of multivalued logic has been discussed.

In the past few years some work has also been initiated in the field of optoelectronic cellular automata for massive parallel processing tasks in relation to parallel computing [6-7] as well as for parallel processor for vision tasks [8]. Optics has advantages over electronics not only in parallel processing but also in four- bit representation. As in this case along with presence or absence of light the nature of the polarization state of the light beam may be an additional parameter for representing such states.

2. Multivalued Logic

2.1 Galois field of the form GF(2^m)

The field of integers modulo a prime number is, of course,

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the most familiar example of a finite field, but many of its properties extend to arbitrary finite fields. Let F be a field . A subset p of that is itself a field under the operations of F will be called a sub field of F and F is called an extension (field) of p. If $p \neq F$, it is said that p is a proper subfield of F. If F is a finite field such that F has p^m elements, where prime p is the characteristic of F and m is the degree of F over its prime subfield then the finite (Galois) field $GF(p^m)$, which has p^m elements, supports basic arithmetic operations under the closure condition. For quaternary logic i.e, with four states, Galoi's field may be represented as $GF(2^2)$.

However, the data representation in such cases is important. The elements of $GF(p^{2m})$ can be represented using the integers from 0 to p-1 , where p is at least two but $GF(2^m)$, the extended field can not be represented by integers alone in a straight forward way. The main conventions for representing elements of $GF(p^m)$ are the exponential format and the polynomial format. Here we present a method for constructing the Galoi's field of 2^m elements (m>1) from the binary field GF(2), (m>1) from the binary field GF(2). The basis of this representation is if all $\gamma_i \in GF(2)$ for $0 \leq i \leq m-1$, are linearly independent, over the set $\Gamma = \{ \gamma_0, \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{m-1} \}$ form a basis of $GF(2^m)$ can be expressed as $\alpha = \Sigma a_i \gamma_i$, where $a_i \in GF(2)$. The term a_i is represented as the i^{th} co-ordinate of a with respect to basis Γ .

Starting with two elements, 0 and 1 from GF(2) and a new symbol α representation of the Galois field for $GF(2^m)$ may be made so that we have the following set of elements on which a multiplication operation "**.**" is defined :

$$F = \{ 0, 1, \alpha , \alpha^2, \ldots\ldots\ldots\ \alpha^j, \ldots\ldots\}$$

the element 1 is α^0 [9].

Next we put a condition on element α so that the set F contains only 2^m elements and is closed under the multiplication "**.**" Let p(x) be a primitive polynomial of degree m over GF(2). We assume that $P(\alpha) =0$. Since p(x) divides $x^{2^m-1} + 1$, we have

$$x^{2^m-1} + 1 = q(x)\, p(x) \quad \ldots\ldots\ldots\ldots\ldots (3)$$

If we replace x by α, we obtain

$$\alpha^{2^m-1} + 1 = q(\alpha)\, p(\alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Since, $p(\alpha) = 0$, we have

$$\alpha^{2^m-1} + 1 = 0$$

Adding 1 to both sides (using modulo 2 addition), we obtain

$$\alpha^{2^m-1} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

So, under this condition that $p(\alpha)=0$, the set F become finite and contains the following elements:

$$F^* = \{0, 1, \alpha, \alpha^2, \ldots\ldots\ldots, \alpha^{2^m-2}\}$$

The non-zero elements of F* are closed under the multiplication operation defined by (1). Let i and j be two integers such that, $0 \leq i$, $j < 2^{m-1}$. If $i + j < 2^{m-1}$ then

$$\alpha^i \,.\, \alpha^j = \alpha^{i+j} ,$$

which is obviously a non-zero element in F. If $i + j \geq 2^{m-1}$, we can express

$$i + j =(2^m-1) + r, \quad \text{where } 0 \leq r < 2^{m-1} .$$

Then $\alpha^i . \alpha^j = \alpha^{i+j} = \alpha^{(2^m-1) + r} = \alpha^{2^{m-1}} . \alpha^r = 1 . \alpha^r = \alpha^r$

which is also a non-zero element in F*. Hence, it can be concluded that the non-zero elements are closed under the multiplication "**.**" .

Also we see that for $0< i < 2^{m-1}$, $\alpha^{2^m - i - 1}$ is the inverse of α^i since

$$\alpha^{2^{m-i-1}} \,.\, \alpha^i = \alpha^{2^m - 1} = 1$$

Hence, $\{ 1, \alpha , \alpha^2, \ldots\ldots\ldots\ \alpha^{2^m-2} \}$ represent $2^m - 1$ distinct elements.

Next we define an additive operation + on F* so that F* forms a commutative group under "+". For $0 \leq i \leq 2^m - 1$ we divide the polynomial x^i by p(x) and obtain the following

$$x^i = q_i(x)p(x) + Q_i(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

where $q_i(x)$ and $Q_i(x)$ are the quotient and the remainder respectively. The remainder $Q_i(x)$ is obviously a polynomial of degree m-1 or less over GF(2) and is of the form

$$Q_i(x) = Q_{i0} + Q_{i1}x + Q_{i2}x^2 + \ldots\ldots\ldots + Q_{i,\, m-1}x^{m-1} \quad \ldots\ldots\ldots(7)$$

Since x and p(x) are relatively prime i.e., they do not have any common factor except 1, x^i is not divisible by p(x). Thus elements in the $GF(2^m)$ are $\{ 0, 1, \alpha , \alpha^2, \ldots\ldots\ldots \alpha^{2^m-2} \}$ where α is the root of m degree primitive polynomial.

If m=2 then $2^m - 2 = 2$.

Hence $\{0,1, \alpha, \alpha^2 \} \equiv \{ 0,1, \alpha , \alpha+1 \}$ are the elements in quadruple logic of 2^2 states.

Thus elements of $GF(2^2)$ are

	Ordered pairs
$0 = 0 + 0.\alpha$	0 0
$1 = 0.\alpha + 1$	0 1
$\alpha = \alpha + 1.0$	1 0
$\alpha^2 = \alpha + 1$	1 1

Ordered pairs represent the elements and the states {0,1,2,3} are represented by dibit as {00, 01, 10, 11} respectively.

2.2 Truth Tables based on dibit representation

The basic logical operations with dibit representation as mentioned in the Sec.2.1 may be expressed in the following