Chapter 6

Fibring first-order logics

So far we have considered propositional based logics. However, sometimes, propositional based logics are not expressive enough for our purposes. We have then to work with more expressive logics, such as first-order or even higher order logics. Clearly, fibring mechanisms are still useful in these settings. We postpone the study of fibring higher-order logics to Chapter 7 and concentrate herein in the study of fibring first-order based logics.

In the first-order setting signatures may include connectives, as in propositional signatures, but they may also include function symbols, predicate symbols and quantifiers (variable binding operators). Two kinds of variables are now involved in the language: schema variables, as before, and quantification variables, that is, the variables that can be bind by the quantifiers.

From the deductive point of view, Hilbert calculi are adopted. The presence of quantifiers in first-order languages introduces some new problems. In particular, the substitution of schema variables within the scope of quantifiers may have unexpected and undesirable consequences. Hence, substitutions have to be carefully handled when using inference rules in a derivation. To deal with this problem we introduce the notion of proviso. Each inference rule includes a proviso whose purpose is to ensure a safe use of substitutions when the rule is applied.

With respect to semantics, things also become more elaborate in this first-order setting. Besides the denotation of connectives, we have to deal with the denotation of functions and predicate symbols, as well as the denotation of quantifiers. Therefore, we need semantic structures that, for instance, have to encompass both the semantics of quantifiers and the semantics of modal operators. We have in mind a powerset algebraic semantics recognizing that quantifiers can be seen as modalities. In this perspective, we adopt semantic structures endowed with a set of “points”, a set of assignments to quantification variables and a set of “worlds”, together with maps that associate to each “point” an assignment and a “world”. As a result, quantifiers become special kind of modalities for which assignments...
play the role of worlds. Herein we take a different approach from the one proposed in [168] for studying fusion of first-order modal logics.

In Section 6.1 we introduce first-order based signatures and the corresponding languages. Next, in Section 6.2, we introduce interpretation structures and interpretation systems. First-order Hilbert calculi are presented in Section 6.3. Section 6.4 introduces first-order logic systems. In Section 6.5, we define fibring of first-order based logics. We illustrate the constructions with the case of classical first-order logic and modal classical first-order logic. The preservation by fibring of several methatheorems as well as the preservation of completeness is discussed in Section 6.6. Finally, in Section 6.7, we make some final remarks.

This chapter capitalizes in [242] for the most part. It is also worthwhile to take a look at [241] in [112] as a preliminary investigation on this topic and also to understand better the problems raised by rules with provisos.

6.1 First-order signatures

This section introduces first-order based signatures and the corresponding first-order based languages.

The notion of signature considered in the previous chapters is not rich enough to cope with first-order features. Hence, we have to consider a more sophisticated notion of signature. Besides connectives, first-order based signatures include function and predicate symbols and variable binding operators, usually referred as quantifiers. For technical reasons to be detailed later on, we also include individual symbols as distinct from 0-ary function symbols (constants). Moreover, modalities are herein distinguished from the other connectives.

In what concerns the variables, we have to consider in this setting two kinds of variables. To begin with, we have to consider quantification variables, that is, the variables that quantifiers bind. Then, as before, we consider schema variables. Since first-order based languages include both terms and formulas, we distinguish between term schema variables and formula schema variables. Hence, we assume fixed throughout this chapter the following three denumerable pairwise disjoint sets of variables:

- \( X = \{x_1, x_2, \ldots\} \);
- \( \Theta = \{\theta_1, \theta_2, \ldots\} \);
- \( \Xi = \{\xi_1, \xi_2, \ldots\} \).

The set \( X \) is the set of quantification variables, the set \( \Theta \) is the set of term schema variables and the set \( \Xi \) is the set of formula schema variables.

We also assume as fixed the equality symbol \( \approx \) and the inequality symbol \( \not\approx \).

Next, we introduce the notion of first-order based signature where \( \mathbb{N}^+ \) is the set of all natural numbers greater than 0.