Chapter 7

Fibring higher-order logics

The applications that motivate our approach to combination of logics require sometimes the use of arbitrary higher-order quantifiers, as well as arbitrary modal-like operators. In this chapter, the concept of fibring is extended to higher-order logics endowed with arbitrary modalities and binding operators. The approach used herein is more general than the one introduced in Chapter 6. The generality comes from the categorical semantics where the collection of truth-values is a topos. The transference results are also simpler than in the first-order case. However, one has to pay the price in the sense that this chapter is not easily read without some knowledge of category theory.

This chapter defines a wide class of logic systems equipped with topos semantics and with Hilbert calculi. Since the use of topos theory is fundamental herein, the reader should be acquainted with topos theory and local set theory. Moreover, it is more convenient to present the corresponding notions of fibring as categorical constructions. We refer the reader to [134] for topos theory with a logic flavor and [187] for category theory and topos theory. We also use [18] for local set theory and relationship with higher-order logics. Finally, we refer to [15] for the notion of cocartesian lifting.

It is worth noting that the class of logics to be studied here encompasses many commonly used logics, such as propositional logics, modal logics, quantification logics, typed lambda calculi and higher-order logics. Arbitrary modalities and binding operators are allowed, as well as any choice of rigid (world-independent) as well as flexible (world-dependent) function symbols.

The deduction mechanism considered herein, as in other chapters, is the Hilbert calculus style, but allowing in this case rules with provisos.

In what concerns semantics, the structures considered in this chapter generalize the usual topos semantics of higher-order logic. This generalization, while preserving the simplicity and elegance of the traditional topos semantics, is able to deal with arbitrary modalities, quantifiers and other binding operators. As done in other chapters, two entailment relations are defined: the local entailment
as usually considered in categorical logic, and the global entailment necessary to deal with necessitation and generalization. Examples are given of familiar logics for which it is possible to lift the original semantics to the topos semantics level, while preserving the denotation of terms (and formulas). Thus, there is no loss of generality by assuming that the logics under consideration are endowed with the kind of topos semantics proposed here.

With respect to soundness, the novelty here is that the usual notion of soundness must be modified in the present framework. This is a consequence of the possibility of having empty domains interpreting the types (a basic feature of categorical semantics). It is proved that the basic example of HOL (a Hilbert-style axiomatization of intuitionistic higher-order logic) is sound with respect to a slightly generalized notion of topos semantics.

We establish a general completeness theorem about full logic systems: every full logic system with Hilbert calculus, including HOL, and enjoying the metatheorem of deduction, is complete. To prove this result we show that every consistent Hilbert calculus that includes HOL and enjoys the metatheorem of deduction has a canonical model. The construction of the canonical model is done as usual in categorical logic (see for instance [18]), but with the adaptations made necessary in view of the richer language we work with (arbitrary modalities and binding operators), and also in view of the two notions of entailment. This theorem plays an important role in the proof of the preservation of completeness by fibring. We first prove that the fibring of full logic systems endowed with Hilbert calculi that include HOL and with the metatheorem of deduction is also complete. Additionally, we show that, under some natural conditions, a full logic system is complete if and only if it can be conservatively enriched with HOL. Finally, as a consequence of this result, a second completeness preservation result is obtained: the fibring of two full, complete logic systems is also complete provided that conservativeness of HOL-enrichment is preserved. It is an open problem to find sufficient conditions for the preservation of HOL-enrichment.

This chapter is structured as follows. In Section 7.1 we introduce the relevant signatures. Section 7.2 presents the Hilbert calculus. Section 7.3 is dedicated to setting up the semantic notions. Section 7.4 introduces the notion of logic system, and we briefly discuss the related notions of soundness and completeness. In Section 7.5, a general completeness theorem is established. In Section 7.6, the notions of constrained and unconstrained fibring of logic systems are given, and it is shown that soundness is preserved by fibring and a completeness preservation result is obtained. Finally, in Section 7.7, we briefly discuss the main results described in the chapter, as well as some open problems related to the completeness preservation by fibring.

The contents of this chapter is based on [62].