Robust Control PID for Time Delays Systems

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Abstract—In this paper we present a robust stability analysis for a time-delay systems, in closed loop with proportional, proportional integral and proportional derivative (P, PI and PD respectively) controllers. The frequency approach is used in order to obtain necessary and sufficient conditions for the robust stability of the characteristic equation in closed loop and to tune the controller. The time domain analysis is used to obtain sufficient stability conditions for the system in closed loop with the PI controller, with nonlinear uncertainties. Also, the results are applied to a binary column distillation and coupled tanks, process commonly used in industry applications.

I. INTRODUCTION

Time delay systems are frequently found in the industry, such as manufacturing system, turbojet engine, telecommunication, economic systems and chemical engineering systems. Therefore, the robust stability analysis for time delay systems has been widely achieved in the last decades [7]. For time delay systems when a simple control law is introduced, it may cause undesirable changes in the behavior of the system, such as instability and oscillations [7]. Besides, if the system has nonlinear uncertainties of non-model dynamics, then a chaotic behavior may occur [3]. Therefore, the study of stability regions is an interesting problem. On the other hand, the main goal in this contribution is to apply the analysis of robust stability to two systems widely used in industry: column distillation and coupled tanks.

Distillation is the most common unit operation in the chemical industry [8]. It is well-known that dynamic nonlinearities, control loop interactions, and system directionality can make the dual composition control in distillation columns a challenging problem. These issues can be effectively tackled by using advanced control systems, but in some cases the implementation of advanced controller are difficult to understand and expensive to maintain [1]. Then the PI controller is the most useful controller. However there does not exist an analysis of robust stability for a distillation plant. In the case of coupled tanks, these relate to fluid transport and liquid level control problems as they would typically occur in process control industries.

The paper is organized as follows: in Section II the problem statement is presented. The frequency domain analysis is presented in Section III. Robust stability using Linear Matrix Inequality (LMI) approach is given in Section IV. Section V introduces simulation result of the illustrative examples, the binary column distillation and the simulation results of two coupled tanks respectively. Concluding remarks are given in Section VI.

II. PROBLEM FORMULATION

Let us consider the following time delay system given in a transfer function form:

\[
\frac{Y(s)}{U(s)} = \frac{Ke^{-sh}}{Ts + 1}
\]  

(1)

When the input signal \(U(s)\) is a PI controller, two gains must be determined: \(K_p\) and \(K_i\). Some methods exist in order to determine these gains [5] but only a few works exist that study the robust stability analysis for parameters uncertainties or nonlinear uncertainties. Besides, since the PI controller has an integral part, the time domain analysis it is involved, then, a transformation on the space state representation it is introduced.

III. FREQUENCY DOMAIN ANALYSIS

In order to synthesize a PI controller for system (1) \(D\)-partitions method [4] is used. Considering that \(U(s)\) is a proportional controller, the characteristic equation in closed loop is given by

\[
Ts + 1 + K_pKe^{-sh}
\]

Now, according with the \(D\)-partitions method, the first boundary of the hyper surfaces is given when \(s = 0\)

\[
1 + K_pK = 0.
\]

(2)

Therefore,

\[
K_p = \frac{-1}{K}, \forall K \neq 0.
\]

(3)

Second boundary is given when \(s = j\omega\)

\[
Tj\omega + 1 + K_pK(\cos \alpha h - j \sin \alpha h) = 0.
\]

(4)

It follows that,

\[
1 + K_pK \cos \alpha h = 0
\]

(5)

\[
T\omega - K_pK \sin \alpha h = 0.
\]

(6)

After basic operations we obtain

\[
K_p^* = \frac{\sqrt{1 + T^2\omega^2}}{K^2},
\]

where \(\omega\) is the numeric solution of (4) and (5). Thus, (2) will be stable for any \(K_p \in [K_p^*, K_p^-]\).
Now, we find the stability zone for \( K_I \) with some \( K_p \in [\overline{K}_p, \underline{K}_p] \) fixed. The characteristic equation in closed loop using a PI controller is
\[
Ts^2 + \left(1 + K_f K_e^{-\omega h}\right) + K_1 Ke^{-\omega h}.
\] (7)
The first boundary \((s = 0)\) is
\[
K_I = 0, \quad \forall K \neq 0.
\]
The second boundary \((s = \omega h)\) for \( K_I \) is given by
\[
-\omega^2 + KK_f \cos \omega h + KK_p \omega \sin \omega h = 0
\]
\[
-\omega + KK_f \sin \omega h - KK_p \omega \sin \omega h = 0.
\] (9)

Straightforward operations give
\[
\overline{K}_I = \frac{\omega^2 T^2 + 1 - K_p^2 K^2}{K}.
\] (10)

Where \( \omega \) is the numeric solution of (9). Therefore if \( K_f \in [0, \overline{K}_f] \) and \( K_p \in [\overline{K}_p, \underline{K}_p] \), then the characteristic equation given by (7) will be stable. Now we analyze the robust stability for the parameters \( K \) and \( h \) of the system in closed loop using the proportional controller. Consider the equations given in (4) and (5), making some operations, we arrive to
\[
\omega = \sqrt{\frac{K_p^2 K^2 - 1}{T^2}}.
\] (11)

Rewriting (4) and (5) as
\[
K_p K = \frac{\omega^2 T^2}{\sin \omega h}, \quad K_p K = \frac{1}{\cos \omega h},
\]
It follows that
\[
h = \frac{\tan^{-1}(T \omega)}{\omega}, \quad \omega \neq 0.
\] (12)

Finally, replacing (11) in (12) we obtain
\[
h = \frac{\tan^{-1}\left(\frac{K_p^2 K^2 - 1}{T}\right)}{\sqrt{K_p^2 K^2 - 1}}.
\] (13)

**Remark 1:** It is possible to find different zones of robust stability for the system (1) depending on the gain \( K_p \) values in the controller. Moreover, if the system has multiples controllers, we can obtain the stable values for \( K_p \) and \( K_I \) by tuning one by one, and finding the intersections of the values of \( K_p \) and \( K_I \).

**IV. TIME DOMAIN ANALYSIS**

In this section the robust stability analysis of a time delays system with nonlinear uncertainties and with a PI controller is introduced. From system (1) it is feasible to obtain the following state-space representation
\[
x(t) = Ax(t) + \sum_{i=0}^{m} B_i \overline{u}(t - h_i)
\]
\[
y(t) = C_i x(t) + C_2 d(t)
\] (14)

where \( x(t) \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}, \) \( B_i \in \mathbb{R}^{n \times q_i}, \) \( \overline{u}(t) \in \mathbb{R}^q \) is the input control, \( y(t) \in \mathbb{R}^q \) is the output, \( C_i \in \mathbb{R}^{q \times n}, \)
\[
C_2 \in \mathbb{R}^{q \times n}, \quad d(t) \in \mathbb{R}^n \quad \text{is the vector of disturbance and} \quad m \quad \text{is the number of time delays. Notice that,}
\]
\[
\overline{u}(t) = [u_i(t) \quad u_2(t) \quad K \quad u_3(t)]^T
\]
\[
\tau(t) = [\tau_1(t) \quad \tau_2(t) \quad K \quad \tau_3(t)]^T,
\]
where,
\[
u_k(t) = K_p \epsilon_k(t) + K_k \int_0^t \epsilon_k(t) dt, \quad k = 1, ..., q
\]
\[
e_k(t) = \tau_k(t) - y_k(t), \quad k = 1, ..., q
\]
and rewriting (15) in a matricial form, it follows
\[
\overline{u}(t) = \text{diag} \left[ K_p \epsilon_1(t) \right] + \text{diag} \left[ K_k \right] \int_0^t \tau(t) dt
\]
where,
\[
\tau(t) = [\epsilon(t) \quad K \quad \epsilon_3(t)]^T
\]

If we choose
\[
x_e(t) = \overline{e}(t)
\]
then
\[
\overline{u}(t) = \overline{K}_p \overline{e}(t) - y(t) + \overline{K}_1 x_e(t)
\]

Thus, (14) can be rewriting as:
\[
x(t) = Ax(t) + \sum_{i=0}^{m} \left[B_i (\overline{K}_p \overline{e}(t - h_i) - \overline{K}_1 C_i x(t - h_i) - \overline{K}_2 C_2 d(t - h_i) + \overline{K}_1 x(t - h_i))\right]
\]
\[
x_e(t) = \overline{e}(t) - y(t) = \overline{e}(t) - C_i x(t) - C_2 d(t)
\] (16)

Without loss of generality, it is assumed that system (14) has an equilibrium point at the origin. When \( r = 0 \), and by considering that \( h_i \) is uncertain, the system has time varying delays bounded by
\[
0 \leq h_i(t) \leq \overline{h} \quad h_i(t) \leq d < 1
\]
where \( \overline{h} \) and \( h_I \) are constants. Then, (16) is rewritten as:
\[
\dot{x} = \sum_{i=0}^{m} \left[\lambda_i (\overline{e}(t - h_i(t))) + \sum_{i=0}^{m} (f_i d(t - h_i(t)))\right]
\] (17)

where
\[
\lambda_i = \begin{bmatrix} \lambda_i(\overline{e}_1(t)) & 0 \end{bmatrix}, \quad f_i = \begin{bmatrix} B_i \overline{K}_1 C_1(\overline{e}_1(t)) & 0 \end{bmatrix}
\]
\[
\dot{x}_e = \lambda_i(\overline{e}_2(t)) + \sum_{i=0}^{m} (f_i d(t - h_i(t)))
\]

where