Abstract. This paper presents the formulation of energy-dissipative momentum-conserving time-stepping algorithms for finite strain dynamic plasticity. These methods require special return mapping algorithms for the integration of the plastic evolution equations, as well as the proper assumed strain treatment to arrive at fully conserving, locking-free assumed strain B-bar finite element methods.

Key words: dynamic plasticity, energy-dissipative momentum-conserving time-stepping algorithms, assumed strain finite element methods.

1 Introduction

Classical time-stepping algorithms like Newmark, HHT and similar are known to develop numerical instabilities in the geometrically nonlinear range. The instabilities are characterized by an unlimited growth of energy [12], and occur even when the same scheme is shown to be unconditionally stable in the linear range. This situation has motivated the development of time-stepping algorithms that exactly conserve the energy for nonlinear elasticity, as presented in [5, 6, 12] among others. The conservation of linear and angular momenta, and the associated relative equilibria [7], is also of the main interest in this nonlinear range, as it is the incorporation of a controllable high-frequency energy dissipation to handle the high numerical stiffness of the mechanical systems of interest; see [3] for recent developments along these lines.

The same numerical instabilities have been observed in the elastoplastic range for existing classical schemes. Despite the dissipative character of the physical system, the energy evolution in the discrete problem is not monotonic leading also to its unstable growth. Algorithms that avoid this situation have been presented in [9, 10]. We present here a summary of the results presented in [1, 4] identifying a new return mapping algorithm for the integration of the plastic evolution equations that leads to the exact energy dissipation and to global energy-dissipative momentum-conserving (EDMC) schemes. Elastoplastic problems require the consideration of locking-free
assumed strain B-bar finite elements. As shown in [2], standard treatments like the one presented in [13] for static problems destroy the conservation properties gained by the conserving temporal integration. This motivates the development of an alternative conserving assumed strain treatment as also presented in this paper.

2 The Continuum Problem

We summarize briefly in this section the equations governing the motion of an elastoplastic solid in the finite deformation range, and the characterization of the momentum conservation laws and energy dissipation along these motions.

2.1 The Equation of Motion and the Momentum Conservation Laws

The aim is to solve for the motion \( \varphi(X,t) \in \mathbb{R}^3 \) of a solid with reference placement \( X \in B \subset \mathbb{R}^3 \) in time \( t \), when subjected to an external body force \( \rho_o b \) (with reference density \( \rho_o \)), external tractions \( \overline{T} \) on the part of the boundary \( \partial T B \) and an imposed deformation \( \varphi = \bar{\varphi} \) on the complementary part \( \partial \varphi B \). The equation describing this motion reads in weak form

\[
\int_B \rho_o \ddot{\varphi} \cdot \eta \, dV + \int_B S : \left( F^T \text{GRAD} \left[ \eta \right] \right) \, dV = \int_B \rho_o b \cdot \eta \, dV + \int_{\partial_T B} \bar{T} \cdot \eta \, dA , \tag{1}
\]

for all admissible variations \( \eta(X) \in \mathbb{R}^3 \), that is, vanishing on \( \partial \varphi B \). Here we have introduced the material acceleration \( \ddot{\varphi} := \left( \frac{\partial^2 \varphi}{\partial t^2} \right) \bigg|_X \), the deformation gradient \( F = \text{GRAD} \left[ \varphi \right] := \left( \frac{\partial \varphi}{\partial X} \right) \bigg|_t \), with the right Cauchy–Green tensor \( C = F^T F \), and the symmetric second Piola–Kirchhoff stress tensor \( S \) given by the constitutive relations discussed below. We point out the notation \( \delta C(\varphi, \eta) \) for the indicated combination of those two functions as arguments.

The structure of equation (1) leads directly to the conservation of the linear \( l \) and angular \( j \) momenta, defined in terms of the material velocity \( V = \dot{\varphi} := \left( \frac{\partial \varphi}{\partial t} \right) \bigg|_X \) as

\[
l := \int_B \rho_o V \, dV = \text{constant} , \quad \text{and} \quad j := \int_B \rho_o \varphi \times V \, dV = \text{constant} , \tag{2}
\]

along the motions with, say, no external loading (\( \rho b = 0, \bar{T} = 0 \) and \( \partial \varphi B = \emptyset \)). These properties follow from the vanishing of the stress term in (1) for the variations corresponding to a translation and infinitesimal rotation, that is,