If \( \{X_t\} \) is a real-valued stationary process, then from a second-order point of view it is characterized by its mean \( \mu \) and its autocovariance function \( \gamma(\cdot) \). The estimation of \( \mu \), \( \gamma(\cdot) \) and the autocorrelation function \( \rho(\cdot) = \gamma(\cdot)/\gamma(0) \) from observations of \( X_1, \ldots, X_n \), therefore plays a crucial role in problems of inference and in particular in the problem of constructing an appropriate model for the data. In this chapter we consider several estimators which will be used and examine some of their properties.

\section*{§7.1 Estimation of \( \mu \)}

A natural unbiased estimator of the mean \( \mu \) of the stationary process \( \{X_t\} \) is the sample mean

\[ \bar{X}_n = n^{-1}(X_1 + X_2 + \cdots + X_n), \]  

(7.1.1)

We first examine the behavior of the mean squared error \( E(\bar{X}_n - \mu)^2 \) for large \( n \).

\begin{theorem}
If \( \{X_t\} \) is stationary with mean \( \mu \) and autocovariance function \( \gamma(\cdot) \), then as \( n \to \infty \),

\[ \text{Var}(\bar{X}_n) = E(\bar{X}_n - \mu)^2 \to 0 \quad \text{if} \quad \gamma(n) \to 0, \]

and

\[ nE(\bar{X}_n - \mu)^2 \to \sum_{h=-\infty}^{\infty} \gamma(h) \quad \text{if} \quad \sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty. \]
\end{theorem}
§7.1. Estimation of \( \mu \)

**Proof.**

\[
\begin{align*}
    n \text{Var}(\bar{X}_n) &= \frac{1}{n} \sum_{i,j=1}^{n} \text{Cov}(X_i, X_j) \\
    &= \sum_{|h|<n} \left( 1 - \frac{|h|}{n} \right) \gamma(h) \\
    &\leq \sum_{|h|<n} |\gamma(h)|.
\end{align*}
\]

If \( \gamma(n) \to 0 \) as \( n \to \infty \) then \( \lim_{n \to \infty} n^{-1} \sum_{|h|<n} \gamma(h) = 2 \lim_{n \to \infty} |\gamma(n)| = 0 \), whence \( \text{Var}(\bar{X}_n) \to 0 \). If \( \sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty \) then the dominated convergence theorem gives

\[
    \lim_{n \to \infty} n \text{Var}(\bar{X}_n) = \lim_{n \to \infty} \sum_{|h|<n} \left( 1 - \frac{|h|}{n} \right) \gamma(h) = \sum_{h=-\infty}^{\infty} \gamma(h). \tag*{\Box}
\]

**Remark 1.** If \( \sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty \), then \( \{X_i\} \) has a spectral density \( f(\cdot) \) and, by Corollary 4.3.2,

\[
    n \text{Var}(\bar{X}_n) \to \sum_{h=-\infty}^{\infty} \gamma(h) = 2\pi f(0).
\]

**Remark 2.** If \( X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} \) with \( \sum_{j=-\infty}^{\infty} |\psi_j| < \infty \), then \( \sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty \) (see Problem 3.9) and

\[
    n \text{Var}(\bar{X}_n) \to \sum_{h=-\infty}^{\infty} \gamma(h) = 2\pi f(0) = \sigma^2 \left( \sum_{j=-\infty}^{\infty} |\psi_j| \right)^2.
\]

**Remark 3.** Theorem 7.1.1 shows that if \( \gamma(n) \to 0 \) as \( n \to \infty \), then \( \bar{X}_n \) converges in mean square (and hence in probability) to the mean \( \mu \). Moreover under the stronger condition \( \sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty \) (which is satisfied by all ARMA\((p,q)\) processes) \( \text{Var}(\bar{X}_n) \sim n^{-1} \sum_{h=-\infty}^{\infty} \gamma(h) \). This suggests that under suitable conditions it might be true that \( X_n \) is AN\((\mu, n^{-1} \sum_{h=-\infty}^{\infty} \gamma(h)) \). One set of assumptions which guarantees the asymptotic normality is given in the next theorem.

**Theorem 7.1.2.** If \( \{X_i\} \) is the stationary process,

\[
    X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \quad \{Z_i\} \sim \text{IID}(0, \sigma^2),
\]

where \( \sum_{j=-\infty}^{\infty} |\psi_j| < \infty \) and \( \sum_{j=-\infty}^{\infty} \psi_j \neq 0 \), then

\[
    \bar{X}_n \text{ is } \text{AN}(\mu, n^{-1} v),
\]

where \( v = \sum_{h=-\infty}^{\infty} \gamma(h) = \sigma^2 \left( \sum_{j=-\infty}^{\infty} |\psi_j| \right)^2 \), and \( \gamma(\cdot) \) is the autocovariance function of \( \{X_i\} \).

**Proof.** See Section 7.3. \( \Box \)