Chapter 6
One-sample problem: Spatial sign test and spatial median

Abstract The spatial sign score function \( U(y) \) is used for the one-sample location problem. The test is then the spatial sign test, and the estimate is the spatial median. The tests and estimates using outer standardization as well as those using inner standardization are discussed.

6.1 Multivariate spatial sign test

6.1.1 Preliminaries

The aim is to find, in the one-sample location problem, test statistics that are valid under much weaker conditions than Hotelling’s \( T^2 \). We consider a multivariate generalization of the univariate sign test, perhaps the simplest test ever proposed. The spatial sign test uses the spatial sign score \( U(y) \) which is given by

\[
U(y) = |y|^{-1}y, \quad \text{for } y \neq 0,
\]

and \( U(0) = 0 \).

We start by giving some approximations and key results needed in the following. Let \( y \neq 0 \) and \( \mu \) be any \( p \)-vectors, \( p > 1 \), and write

\[
r = |y| \quad \text{and} \quad u = |y|^{-1}y.
\]
The accuracies of different (constant, linear and quadratic) approximations of the function $|y - \mu|$ of $\mu$ are given by the following.

**Lemma 6.1.**

1. $||y - \mu| - |y|| \leq |\mu|.$
2. $||y - \mu| - |y| - u'\mu| \leq 2|\mu|^2/r.$
3. $||y - \mu| - |y| - u'\mu - \mu'1/2r(I_p - uu')\mu| \leq C|\mu|^{2+\delta}/r^{1+\delta}$
   
   for all $0 < \delta < 1$ where $C$ does not depend on $y$ or $\mu$.

In a similar way, the accuracies of constant and linear approximations of the function $|y - \mu|^{-1}(y - \mu)$ of $\mu$ are given by

**Lemma 6.2.**

1. $||y - \mu|/|y - \mu| - y/|y|| \leq 2|\mu|/r.$
2. $||y - \mu|/|y - \mu| - y/|y| - 1/2r(I_p - uu')\mu| \leq C|\mu|^{1+\delta}/r^{1+\delta}$
   
   for all $0 < \delta < 1$ where $C$ does not depend on $y$ or $\mu$.

See Appendix B.

We also often need the following lemma.

**Lemma 6.3.** Assume that the density function $f(\varepsilon)$ of the $p$-variate continuous random vector $\varepsilon$ is uniformly bounded. Then $E\{|\varepsilon|^{-\alpha}\}$ exists for all $0 \leq \alpha < 2$.

### 6.1.2 The test outer standardization

We consider first the location model

\[ y_i = \mu + \varepsilon_i, \quad i = 1, ..., n, \]