SOME DISCRETE PROPERTIES OF THE SPACE OF LINE TRANSVERSALS TO DISJOINT BALLS

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Abstract. Attempts to generalize Helly's theorem to sets of lines intersecting convex sets led to a series of results relating the geometry of a family of sets in $\mathbb{R}^d$ to the structure of the space of lines intersecting all of its members. We review recent progress in the special case of disjoint Euclidean balls in $\mathbb{R}^d$, more precisely the inter-related notions of *cone of directions*, *geometric permutations* and *Helly-type theorems*, and discuss some algorithmic applications.

Key words. Geometric transversal, Helly's theorem, line, sphere, geometric permutation, cone of directions.

1. Introduction. Lines intersecting or tangent to prescribed geometric objects are central to various problems in computational geometry and application areas; typical examples include visibility [26, 64] or shortest path [61] computation and robust statistics [14, 67]. To design efficient algorithms for these problems, one first has to understand the geometry of the underlying sets of lines. A natural embedding of the space of lines in $\mathbb{P}^3(\mathbb{R})$ is as a quadric in $\mathbb{P}^5(\mathbb{R})$, the Klein (or Plücker) quadric; in some sense this is optimal[1], so line geometry is, at least in dimension 3, inherently nonlinear.

Let $\mathcal{C}$ be a collection of subsets of $\mathbb{R}^d$, or *objects* for short. Denote by $\mathcal{T}_k(\mathcal{C})$ the set of k-transversals to $\mathcal{C}$, that is of k-dimensional affine subspaces that intersect every member of $\mathcal{C}$. Helly's theorem [42] asserts that if $\mathcal{C}$ consists of convex sets then $\mathcal{T}_0(\mathcal{C})$ is nonempty if and only if $\mathcal{T}_0(F)$ is nonempty for any subset $F \subset \mathcal{C}$ of size at most $d+1$. Whether Helly's theorem generalizes to other values of k is a natural question which was, to my knowledge, first investigated in the 1930's by Vincensini [76]. The answer turns out to be negative in general but positive when the geometry of the objects is adequately constrained. The study of how the geometry of the objects in $\mathcal{C}$ determines the structure of $\mathcal{T}_k(\mathcal{C})$, and subsequent developments of similar flavor, is now designated as *geometric transversal theory* [34].

Helly's theorem was recently generalized to *line transversals* ($k = 1$) to disjoint (Euclidean) balls in $\mathbb{R}^d$, answering in the positive a conjecture of Danzer [23] who settled the 2-dimensional case. This generalization builds on a series of results concerning two notions: *cone of directions* and *geometric permutations*. This survey gives a comprehensive overview of these investigations by presenting, in a unified language, the results of

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[1]Indeed (i) there does not exist any homeomorphism between the lines in $\mathbb{R}^3$ and an open subset of $\mathbb{P}^4(\mathbb{R})$, and (ii) any algebraic homeomorphism between lines in $\mathbb{R}^3$ and points in $\mathbb{P}^5(\mathbb{R})$ has degree at least 2 [65, Remarks 2.1.4 and 2.1.6, p. 143].

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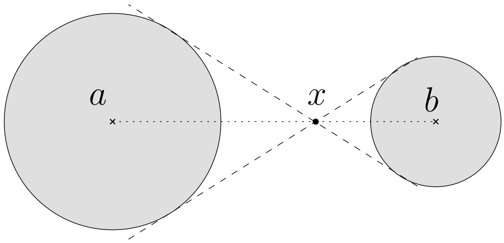

Fig. 1. *The internal center of similitude x of two balls, represented in a 2-plane through their centers.*

several papers [3, 7, 16, 17, 19–21, 45, 48, 53, 73, 81], new extensions of these results and some of their algorithmic consequences. Although some results generalize to other settings, the discussion will focus on the case of line transversals to disjoint balls.

1.1. Notations and terminology. We denote by $\mathbb{R}^d$ the real d-dimensional affine space or, equivalently, the Euclidean d-dimensional space; the Euclidean metric is the only one we consider over $\mathbb{R}^d$. We denote by $\mathbb{P}^d$ the real d-dimensional projective space and by $\mathbb{S}^{d-1}$ the space of directions in $\mathbb{R}^d$, which we identify with the unit sphere. Recall that a *great circle* of $\mathbb{S}^d$ is a section of $\mathbb{S}^d$ by some 2-flat through its center. We write A^o the interior of a set A and use arrows to denote vectors; in particular, we write $\overrightarrow{\ell}$ a direction vector of an oriented line ℓ. We use $\langle \overrightarrow{u}, \overrightarrow{v} \rangle$ and $\angle(\overrightarrow{u}, \overrightarrow{v})$ to denote, respectively, the dot product of and the angle between vectors $\overrightarrow{u}$ and $\overrightarrow{v}$.

A *ball* is closed unless otherwise specified: the ball of center c and radius r is the set of points x such that $|c - x| \leq r$. In particular, *disjoint* balls are not allowed to be tangent; for the sake of simplicity, we say that several balls are *disjoint* if they are pairwise disjoint. A *unit* ball is a ball with radius 1; since transversal properties are unchanged under scaling, results obtained for unit balls usually extend to *congruent* balls, i.e. sets of balls with equal radii. The *radius disparity* of a set of balls is the ratio of the largest radius to the smallest. The *internal center of similitude* of two disjoint balls in $\mathbb{R}^d$ with respective centers a, b and radii r_a, r_b is the point $\frac{r_b a + r_a b}{r_a + r_b}$ (see Figure 1); this point is sometimes referred to as the *geometric center* [81] or the *center of gravity* [47] of the two balls.

We use the terms *collection* or *family* for an unordered set, and *sequence* for an ordered set. We denote by $|X|$ the cardinality of a set X. Given a sequence $\mathcal{C}$, we denote by $\prec_{\mathcal{C}}$ the corresponding ordering on its elements. A *subsequence* of a sequence is a subset of its members, ordered as in the sequence. A *k-transversal* to a collection $\mathcal{C}$ is an affine subspace of dimension k that intersects every member of $\mathcal{C}$; for the sake of simplicity, we say *transversal* for 1-transversal, that is line transversal, and speak