Finite Rank Toeplitz Operators in the Bergman Space

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Abstract We discuss resent developments in the problem of description of finite rank Toeplitz operators in different Bergman spaces and give some applications.

1 Introduction

Toeplitz operators arise in many fields of Analysis and have been an object of active study for many years. Quite a lot of questions can be asked about these operators, and these questions depend on the field where Toeplitz operators are applied.

The classical Toeplitz operator $T_f$ in the Hardy space $H^2(S^1)$ is defined as

$$T_fu = Pf u,$$

(1.1)

for $u \in H^2(S^1)$, where $f$ is a bounded function on $S^1$ (the weight function) and $P$ is the Riesz projection, the orthogonal projection $P : L_2(S^1) \to H^2(S^1)$. Such operators are often called Riesz–Toeplitz or Hardy–Toeplitz operators (cf. [15], for more details). More generally, for a Hilbert space $\mathcal{H}$ of functions and a closed subspace $\mathcal{L} \subset \mathcal{H}$, the Toeplitz operator $T_f$ in $\mathcal{L}$ acts as in (1.1), where $P$ is the projection $P : \mathcal{H} \to \mathcal{L}$. In particular, in the case where $\mathcal{H}$ is the space $L_2(\Omega, \rho)$ for some domain $\Omega \subset \mathbb{C}^d$ and some measure $\rho$
and $\mathcal{L}$ is the Bergman space $\mathcal{B}^2 = \mathcal{B}^2(\Omega, \rho)$ of analytical functions in $\mathcal{H}$, such an operator is called Bergman–Toeplitz; we denote it by $T_f$.

Among many interesting properties of Riesz–Toeplitz operators, we mention the following cut-off one. If $f$ is a bounded function and the operator $T_f$ is compact, then $f$ should be zero. For many other classes of operators a similar cut-off on some level is also observed. The natural question arises, whether there is a kind of cut-off property for Bergman–Toeplitz operators. Quite long ago it became a common knowledge that at least direct analogy does not take place. In [13], the conditions were found on the function $f$ in the unit disk $\Omega = D$ guaranteeing that the operator $T_f$ in $\mathcal{B}^2(D, \lambda)$ with Lebesgue measure $\lambda$ belongs to the Schatten class $\mathcal{S}_p$. So, the natural question came up: probably, it is on the finite rank level that the cut-off takes place. In other words, if a Bergman–Toeplitz operator has finite rank it should be zero.

It was known long ago that the Schatten class behavior of $T_f$ is determined by the rate of convergence to zero at the boundary of the function $f$. Therefore, the finite rank (FR) hypothesis deals with functions $f$ with compact support not touching the boundary of $\Omega$. In this setting, the FR hypothesis is equivalent to the one for Toeplitz operators on the Bargmann (Fock, Segal) space consisting of analytical functions in $\mathbb{C}$, square summable with a Gaussian weight. A proof of the FR hypothesis appeared in the same paper [13], about twenty lines long. Unfortunately, there was an unrepairable fault in the proof, so the FR remained unsettled.

It was only in 2007 that the proof of the FR hypothesis was finally found, even in a more general form. The Bergman projection $P : L_2 \to \mathcal{B}$ can be extended to an operator from the space of distributions $\mathcal{D}'(\Omega)$ to $\mathcal{B}^2(\Omega, \lambda)$. Let $\mu$ be a regular complex Borel measure with compact support in $\Omega$. With $\mu$ we associate the Toeplitz operator $T_\mu : u \mapsto Pu\mu$ in $\mathcal{B}^2(\Omega, \lambda)$.

In [14], the following result was established.

**Theorem 1.1.** Suppose that the Toeplitz operator $T_\mu$ in $\mathcal{B}^2(\Omega, \lambda)$, $\Omega \subset \mathbb{C}$ has finite rank $r$. Then the measure $\mu$ is the sum of $r$ point masses,

$$\mu = \sum_{1}^{r} C_k \delta_{z_j}, \ z_j \in \Omega. \quad (1.2)$$

The publication of the proof of Theorem 1.1 induced an activity around it. In two years to follow several papers appeared, where the FR theorem was generalized in different directions, and interesting applications were found in Analysis and Mathematical Physics.

In this paper, we aim for collecting and systematizing the existing results on the finite rank problem and their applications. We also present several new theorems generalizing and extending these results.

In a more vague setting, the problem discussed in the paper can be understood in the following way: is it possible that the contribution of the positive