Estimates for the Counting Function of the Laplace Operator on Domains with Rough Boundaries

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Abstract We present explicit estimates for the remainder in the Weyl formula for the Laplace operator on a domain \( \Omega \), which involve only the most basic characteristics of \( \Omega \) and hold under minimal assumptions about the boundary \( \partial \Omega \).

This is a survey of results obtained by the authors in the last few years. Most of them were proved or implicitly stated in our papers [10, 11, 12]; we give precise references or outline proofs wherever it is possible. The results announced in Subsection 5.2 are new.

Let \( \Omega \subset \mathbb{R}^n \) be an open bounded domain in \( \mathbb{R}^n \), and let \( -\Delta_B \) be the Laplacian on \( \Omega \) subject to the Dirichlet (\( B = D \)) or Neumann (\( B = N \)) boundary condition. Further on, we use the subscript \( B \) in the cases where the corresponding statement refers to (or result holds for) both the Dirichlet and Neumann Laplacian. Let \( N_B(\Omega, \lambda) \) be the number of eigenvalues of \( \Delta_B \) lying below \( \lambda^2 \). If the number of these eigenvalues is infinite or \( -\Delta_B \) has essential spectrum below \( \lambda^2 \), then we define \( N_N(\Omega, \lambda) := +\infty \). Let

\[
R_B(\Omega, \lambda) := N_B(\Omega, \lambda) - (2\pi)^{-n} \omega_n |\Omega| \lambda^n,
\]

where \( \omega_n \) is the volume of the \( n \)-dimensional unit ball and \( |\Omega| \) denotes the volume of \( \Omega \). According to the Weyl formula, \( R_B(\Omega, \lambda) = o(\lambda^n) \) as \( \lambda \to +\infty \). If \( B = D \), then this is true for every bounded domain \([4]\). If \( B = N \), then the Weyl formula holds only for domains with sufficiently regular boundaries. In

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the general case, $R_N$ may well grow faster than $\lambda^n$; moreover, the Neumann Laplacian on a bounded domain may have a nonempty essential spectrum (see, for instance, Remark 6.1 or [6]). The necessary and sufficient conditions for the absence of the essential spectrum in terms of capacities were obtained by Maz’ya [8].

The aim of this paper is to present estimates for $R_B(\Omega, \lambda)$, which involve only the most basic characteristics of $\Omega$ and constants depending only on the dimension $n$. The estimate from below (1.2) for $R_B(\Omega, \lambda)$ and the estimate from above (4.1) for $R_D(\Omega, \lambda)$ hold for all bounded domains. The upper bound (4.2) for $R_N(\Omega, \lambda)$ is obtained for domains $\Omega$ of class $C^\alpha$, i.e., under the following assumption:

- every point $x \in \partial\Omega$ has a neighborhood $U_x$ such that $\Omega \cap U_x$ coincides (in a suitable coordinate system) with the subgraph of a continuous function $f_x$. If all the functions $f_x$ satisfy the Hölder condition of order $\alpha$, one says that $\Omega$ belongs to the class $C^\alpha$. For domains $\Omega \in C^\alpha$ with $\alpha \in (0, 1)$ our estimates $R_B(\Omega, \lambda) = O(\lambda^{n-\alpha})$ and $R_N(\Omega, \lambda) = O(\lambda^{(n-1)/\alpha})$ are order sharp in the scale $C^\alpha$ as $\lambda \to \infty$. The latter estimate implies that the Weyl formula holds for the Neumann Laplacian whenever $\alpha > 1 - \frac{1}{n}$. If $\alpha \leq 1 - \frac{1}{n}$, then there exist domains in which the Weyl formula for $N_N(\Omega, \lambda)$ fails (see Remark 4.2 for details or [11] for more advanced results).

For domains of class $C^\infty$ our methods only give the known remainder estimate $R_B(\Omega, \lambda) = O(\lambda^{n-1} \log \lambda)$. To obtain the order sharp estimate $O(\lambda^{n-1})$, one has to use more sophisticated techniques. The most advanced results in this direction were obtained in [7], where the estimate $R_B(\Omega, \lambda) = O(\lambda^{n-1})$ was established for domains which belong to a slightly better class than $C^1$.

Throughout the paper, we use the following notation.

- $d(x)$ is the Euclidean distance from the point $x \in \Omega$ to the boundary $\partial\Omega$;
- $\Omega_\delta^b := \{x \in \Omega \mid d(x) \leq \delta\}$ is the internal closed $\delta$-neighborhood of $\partial\Omega$;
- $\Omega_\delta := \Omega \setminus \Omega_\delta^b$ is the interior part of $\Omega$.

1 Lower Bounds

Denote by $\Pi_B(\lambda)$ the spectral projection of the operator $-\Delta_B$ corresponding to the interval $[0, \lambda^2]$. Let $e_B(x, y; \lambda)$ be its integral kernel (the so-called spectral function). It is well known that $e_B(x, y; \lambda)$ is an infinitely differentiable function on $\Omega \times \Omega$ for each fixed $\lambda$ and that $e_B(x, x; \lambda)$ is a nondecreasing polynomially bounded function of $\lambda$ for each fixed $x \in \Omega$.

By the spectral theorem, the cosine Fourier transform of $\frac{d}{d\lambda} e_B(x, y; \lambda)$ coincides with the fundamental solution $u_B(x, y; t)$ of the wave equation in $\Omega$. On the other hand, due to the finite speed of propagation, $u_B(x, x; t)$ is equal to $u_0(x, x; t)$ whenever $t \in (-d(x), d(x))$, where $u_0(x, y; t)$ is the