A 2-CATEGORIES COMPANION

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Abstract. This paper is a rather informal guide to some of the basic theory of 2-categories and bicategories, including notions of limit and colimit, 2-dimensional universal algebra, formal category theory, and nerves of bicategories.

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1. Overview and basic examples. This paper is a rather informal guide to some of the basic theory of 2-categories and bicategories, including notions of limit and colimit, 2-dimensional universal algebra, formal category theory, and nerves of bicategories. As is the way of these things, the choice of topics is somewhat personal. No attempt is made at either rigour or completeness. Nor is it completely introductory: you will not find a definition of bicategory; but then nor will you really need one to read it. In keeping with the philosophy of category theory, the morphisms between bicategories play more of a role than the bicategories themselves.

1.1. The key players. There are bicategories, 2-categories, and **Cat**-categories. The latter two are exactly the same (except that strictly speaking a **Cat**-category should have small hom-categories, but that need not concern us here). The first two are nominally different — the 2-categories are the strict bicategories, and not every bicategory is strict — but every bicategory is *biequivalent* to a strict one, and biequivalence is the right general notion of equivalence for bicategories and for 2-categories. Nonetheless, the theories of bicategories, 2-categories, and **Cat**-categories have rather different flavours.

An enriched category is a category in which the hom-functors take their values not in **Set**, but in some other category $\mathscr{V}$. The theory of enriched categories is now very well developed, and **Cat**-category theory is the special case where $\mathscr{V} = \mathbf{Cat}$. In **Cat**-category theory one deals with higher-dimensional versions of the usual notions of functor, limit, monad, and so on, without any "weakening." The passage from category theory to **Cat**-category theory is well understood; unfortunately **Cat**-category theory is generally not what one wants to do — it is too strict, and fails to deal with the notions that arise in practice.

In bicategory theory all of these notions are weakened. One never says that arrows are equal, only isomorphic, or even sometimes only that there is a comparison 2-cell between them. If one wishes to generalize a

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result about categories to bicategories, it is generally clear in principle what should be done, but the details can be technically very difficult.

2-category theory is a "middle way" between **Cat**-category theory and bicategory theory. It *uses* enriched category theory, but not in the simple minded way of **Cat**-category theory; and it cuts through some of the technical nightmares of bicategories. The prefix "2-," as in 2-functor or 2-limit, will always denote the strict notion; although often we will use it to describe or analyze non-strict phenomena.

There are also various other related notions, which will be less important in this companion. **SSet**-categories are categories enriched in simplicial sets; every 2-category induces an **SSet**-category, by taking nerves of the hom-categories. Double categories are internal categories in **Cat**. Once again every 2-category can be seen as a double category. A slight generalization of double categories allows bicategories to fit into this picture. Finally there are the internal categories in **SSet**; both **SSet**-categories and double categories can be seen as special cases of these.

1.2. Nomenclature and symbols. In keeping with our general policy, the word *2-functor* is understood in the strict sense: a 2-functor between 2-categories $\mathscr{A}$ and $\mathscr{B}$ assigns objects to objects, morphisms to morphisms, and 2-cells to 2-cells, preserving all of the 2-category structure strictly. We shall of course want to consider more general types of morphism between 2-categories later on.

If "widget" is the name of some particular categorical structure, then there are various systems of nomenclature for weak 2-widgets. Typically one speaks of *pseudo* widgets for the up-to-isomorphism notion, *lax* widgets for the up-to-not-necessarily-invertible comparison notion, and when the direction of the comparison is reversed, either *oplax* widget or *colax* widget, depending on the specific case. But there are also other conventions. In contexts where the pseudo notion is most important, this is called simply a widget, and then one speaks explicitly of *strict* widgets in the strict case. In contexts where the lax notion is most important (such as with monoidal functors), it is this which has no prefix; and one has *strict* widgets in the strict case or *strong* widgets in the pseudo.

As we move up to 2-categories and higher categories, there are various notions of sameness, having the following symbols:

- $=$ is equality
- $\cong$ is isomorphism (morphisms f and g with $gf = 1$, $fg = 1$)
- $\simeq$ is equivalence ($gf \cong 1$, $fg \cong 1$)
- $\sim$ is sometimes used for biequivalence.

In Sections 1.4 and 1.5 we look at various examples of 2-categories and bicategories. The separation between the 2-category examples and the bicategory examples is not really about strictness but about the sort of morphisms involved. The 2-category examples involve functions or functors of some sort; the bicategory examples (except the case of a monoidal