Chapter 2

The Probability Space of Brownian Motion

2.1 Introduction

According to Einstein’s description, the Brownian motion can be defined by the following two properties: first, it has continuous trajectories (sample paths) and second, the increments of the paths in disjoint time intervals are independent zero mean Gaussian random variables with variance proportional to the duration of the time interval (it is assumed, for definiteness, that the possible trajectories of a Brownian particle start at the origin). These properties have far-reaching implications about the analytic properties of the Brownian trajectories. It can be shown, for example (see Theorem 2.4.1), that these trajectories are not differentiable at any point with probability 1 [198]. That is, the velocity process of the Brownian motion cannot be defined as a real-valued function, although it can be defined as a distribution (generalized function) [152]. Langevin’s construction does not resolve this difficulty, because it gives rise to a velocity process that is not differentiable so that the acceleration process, \( \dot{\Xi}(t) \) in eq. (1.24), cannot be defined.

One might guess that in order to overcome this difficulty in Langevin’s equation all differential equations could be converted into integral equations so that the equations contain only well defined velocities. This approach, however, fails even in the simplest differential equations that contain the process \( \Xi(t) \) (which in one dimension is denoted \( \Xi(t) \)). For example, if we assume that \( \Delta w(t) \equiv \int_{t}^{t+\Delta t} \Xi(s) \, ds \sim \mathcal{N}(0, \Delta t) \) and construct the solution of the initial value problem

\[
\dot{x} = x \Xi(t), \quad x(0) = x_0 > 0
\]  

(2.1)

by the Euler method

\[
x_{\Delta t}(t + \Delta t) - x_{\Delta t}(t) = x_{\Delta t}(t) \Delta w(t), \quad x_{\Delta t}(0) = x_0 > 0,
\]  

(2.2)
the limit \( x(t) = \lim_{\Delta t \to 0} x_{\Delta t}(t) \) is not the function

\[
x(t) = x_0 \exp \left\{ \int_0^t \Xi(s) \, ds \right\}.
\]

It is shown below that the solution is

\[
x(t) = x_0 \exp \left\{ \int_0^t \Xi(s) \, ds - \frac{1}{2} t \right\}.
\]

It is evident from this example that differential equations that involve the Brownian motion do not obey the rules of the differential and integral calculus.

A similar phenomenon manifests itself in other numerical schemes. Consider, for example, three different numerical schemes for integrating eq. (2.1) (or rather (2.2)), an explicit Euler, semi implicit, and implicit schemes. More specifically, consider the one-dimensional version of eq. (2.2) with \( \Delta w(t) \sim N(0, \Delta t) \). Discretizing time by setting \( t_j = j \Delta t \) for \( j = 0, 1, 2, \ldots \), the random increments \( \Delta w(t_j) = w(t_{j+1}) - w(t_j) \) are simulated by \( \Delta w(t_j) \sim n_j \sqrt{\Delta t} \), where \( n_j \sim N(0, 1) \) are independent (zero mean standard Gaussian random numbers taken from the random number generator). The explicit Euler scheme (2.2) is written as

\[
x_{ex}(t_{j+1}) = x_{ex}(t_j) + x_{ex}(t_j) n_j \sqrt{\Delta t}, \quad \text{with} \quad x_{ex}(0) = x_0 > 0,
\]

the semi implicit scheme is

\[
x_{si}(t_{j+1}) = x_{si}(t_j) + \frac{1}{2} [x_{si}(t_j) + x_{si}(t_{j+1})] n_j \sqrt{\Delta t}, \quad \text{with} \quad x_{si}(0) = x_0 > 0,
\]

and the implicit scheme is

\[
x_{im}(t_{j+1}) = x_{im}(t_j) + x_{im}(t_{j+1}) n_j \sqrt{\Delta t}, \quad \text{with} \quad x_{im}(0) = x_0 > 0.
\]

In the limit \( \Delta t \to 0, t_j \to t \) the numerical solutions converge (in probability) to the three different limits

\[
\lim_{\Delta t \to 0, t_j \to t} x_{ex}(t_j) = x_0 \exp \left\{ \int_0^t \Xi(s) \, ds - \frac{1}{2} t \right\}
\]

\[
\lim_{\Delta t \to 0, t_j \to t} x_{si}(t_j) = x_0 \exp \left\{ \int_0^t \Xi(s) \, ds \right\}
\]

\[
\lim_{\Delta t \to 0, t_j \to t} x_{im}(t_j) = x_0 \exp \left\{ \int_0^t \Xi(s) \, ds + \frac{1}{2} t \right\}.
\]

These examples indicate that naïve applications of elementary analysis and probability theory to the simulation of Brownian motion may lead to conflicting results. The study of the trajectories of the Brownian motion requires a minimal degree of mathematical rigor in the definitions and constructions of the probability space and the probability measure for the Brownian trajectories in order to gain some insight