3 Applications and Extensions

Ordinary differential equations are the backbone of this book. Symbolically this class of problems can be represented by the ODE prototype equation

\[ \dot{y} = f(y) , \]

which is a short way for

\[ \frac{dy(t)}{dt} = f(y(t)) , \tag{3.1} \]

with \( f \) sufficiently smooth. For this type of equation we discuss parameter dependence, bifurcation, and stability in detail. Many other bifurcation problems are not of this ODE type. For example, a delay may be involved, or the dynamics fails to be smooth. But even then the ODE background is helpful. On the one hand, methods can be applied that are similar as the ODE approaches. On the other hand, the ODE system (3.1) can be used to approximate non-ODE situations, or to characterize certain special cases.

This chapter outlines how ODEs play an important role also in non-ODE problems. Several applications and extensions offer insight into other areas. Section 3.1 deals with delay differential equations, Section 3.2 discusses important cases of nonsmooth dynamics, and Section 3.3 introduces some aspects of differential-algebraic equations. Partial differential equations enter with the aspect of pattern formation. This is explained in Section 3.4 with nerve models, and more generally in Section 3.5 for reaction-diffusion problems. Finally, in Section 3.6, we point out the importance of the bifurcation machinery for deterministic risk analysis.

3.1 Delay Differential Equations

In (3.1) the derivative \( \dot{y} = f(y) \) is given as a function of \( y(t) \)—that is, the derivative and \( f(y) \) are both evaluated at time \( t \). For numerous applications this dependence of \( f \) is not adequate. Rather, the derivative \( \dot{y} \) may depend on the state \( y(t-\tau) \) at an earlier time instant \( t-\tau \). The past time is specified by the delay \( \tau \). The delay differential equation is then \( \dot{y}(t) = f(y(t-\tau)) \), or more general,

\[ \dot{y}(t) = f(t, y(t), y(t-\tau)) \tag{3.2} \]
Whereas initial-value problems of ODEs just need one initial vector at $t_0$, delay equations require an initial function $\phi$ on an interval, 

$$y(t) = \phi(t) \quad \text{for} \quad t_0 - \tau \leq t \leq t_0.$$ 

Delay equations are used to model many systems, in particular in engineering, biology, and economy.

A nice example of a system where delay occurs naturally is the production of red blood cells in the bone marrow. It takes about four days for a new blood cell to mature. When $b(t)$ represents the number of red blood cells, then their death rate is proportional to $-b(t)$, whereas the reproduction rate depends on $b(t - \tau)$ with a delay $\tau$ of about four days. Two pioneering models are given in [WaL76], [MaG77]. Here we analyze the model of [WaL76].

**Example 3.1 Production of Blood Cells**

For positive $\delta, p, \nu, \tau$ the scalar delay differential equation

$$\dot{b}(t) = -\delta b(t) + p e^{-\nu b(t-\tau)} \quad (3.3)$$

serves as model for production of blood cells in the bone marrow. The first term on the right-hand side is the decay, with $0 < \delta < 1$ because $\delta$ is the probability of death. The second term describes the production of new cells with a nonlinear rate depending on the earlier state $b(t - \tau)$. The equation has an equilibrium $b^s$, given by $\delta b^s = p \exp(-\nu b^s)$. A Newton iteration for

$$\delta = 0.5, \; p = 2, \; \nu = 3.694$$

gives the equilibrium value $b^s = 0.5414$. The model (3.3) exhibits a Hopf bifurcation off the stationary solution $b^s$. We postpone a discussion of the stability of $b^s$, and start with a transformation to a framework familiar to us from Section 1.4: For an experimental investigation we create an approximating map, based on a discretization by forward Euler (1.17).

**3.1.1 A Simple Discretization**

We stay with Example 3.1. A transformation of (3.3) to normal delay by means of the normalized time $\tilde{t} := t/\tau$ and $u(\tilde{t}) := b(\tau \tilde{t})$ leads to $b(t - \tau) = b(\tau(\tilde{t} - 1)) = u(\tilde{t} - 1)$ and $\frac{du}{dt} = \dot{b} \cdot \tau$, so

$$\frac{du}{d\tilde{t}} = -\delta \tau u + p \tau e^{-\nu u(\tilde{t} - 1)}.$$ 

Introduce a $\tilde{t}$-grid with step size $h = \frac{1}{m}$ for an integer $m$, and denote the approximation to $u(jh)$ by $u_j$. Then the forward Euler approximation is

$$u_{j+1} = u_j - \delta h \tau u_j + ph \tau \exp(-\nu u_{j-m}). \quad (3.4)$$