7 Stability of Periodic Solutions

Periodic solutions of a differential equation are time-dependent or space-dependent orbits, cycles, or oscillations. In this chapter a major concern is time-dependent periodicity ("time periodicity" for short) of solutions of autonomous systems of ODEs

\[
\dot{y} = f(y, \lambda). \tag{7.1}
\]

For time-periodic solutions, there is a minimum time interval \( T > 0 \) (the "period") after which the system returns to its original state:

\[
y(t + T) = y(t)
\]

for all \( t \). In Section 7.7 we turn our attention to the non-autonomous case.

Fig. 7.1. The period \( T \) of a periodic solution

Before discussing time periodicity, let us briefly comment on space-dependent periodicity. Space-periodic phenomena are abundant in nature, ranging from the stripes on a zebra to various rock formations to sand dunes on a beach. PDEs are appropriate means for describing spatial patterns, see Section 3.5. The simple example of the PDE

\[
\nabla \times y + y = 0,
\]

where \( y \) consists of three components \( y_i(x_1, x_2, x_3), \ i = 1, 2, 3, \) depending on three spatial variables \( x_k \) may serve as motivation for spatial periodicity.
7.1 Periodic Solutions of Autonomous Systems

Because the right-hand side $f(y, \lambda)$ of equation (7.1) is autonomous, $y(t + \zeta)$ is a solution of equation (7.1) whenever $y(t)$ is a solution. This holds for all constant phase shifts $\zeta$. Therefore one is free to start measuring the period $T$ of a periodic orbit at any point $y(t_0)$ along the profile (see Figure 7.1). We can fix an “initial” moment $t_0 = 0$ wherever we like; this anchors the profile. Such a normalization is frequently called a phase condition.

A simple phase condition is provided by

$$p(y(0), \lambda) := y_k(0) - \eta = 0.$$

Here, $\eta = \eta(k, \lambda)$ is a prescribed value in the range of $y_k(t)$ [AlC84], [HoK84b]. Choosing $k$ and $\eta$ requires an extra device to determine the current range of $y_k(t)$. This choice of a phase condition has practical disadvantages. Varying $\lambda$, the profile of a periodic solution changes. Fixing $y_k(0) = \eta$ does not prevent that peaks and other maxima and minima drift across the time interval. That is, a change in the profile goes along with a shift in time direction. Such a shift makes changes in the profile harder to judge and requires frequent adaption of the grid of the numerical integration. In view of this situation, it makes sense to request that shifts of the profile are minimal when one passes from one parameter value $\lambda$ to the next. This also allows larger steps during continuation.

There are phase conditions that meet this requirement. One example is given by the relation

$$p(y(0), \lambda) := y_j(0) - f_j(y(0), \lambda) = 0,$$

which demands that $t_0 = 0$ be a critical point of $y_j$. This normalization, proposed in [Sey81a], does not require any adjustment of $j$ or of other quantities. As pointed out in Section 2.9, the corresponding initial value $y_j(0)$ is well suited to be the ordinate of bifurcation diagrams. The index $j$ is arbitrary ($1 \leq j \leq n$); choose $j = 1$ or pick the component of $f$ that has the simplest structure. Equation (7.2) forces a maximum or minimum of $y_j$ to stick at $t = 0$.

Another phase condition is the orthogonality

$$p(y(0), \lambda) := (y(0) - \hat{y}(0))^T f(\hat{y}(0), \lambda) = 0$$

see [Doe81], [Beyn90a]. In (7.3), $\hat{y}$ is a known nearby solution, usually the approximation at the previous parameter value in a continuation. Doedel has suggested the integral phase condition