Numerical Methods in Kinematics

By increasing the number of links, the analytic calculation in robotics becomes a tedious task and numerical calculations are needed. We review the most frequent needed numerical analysis in robotics.

9.1 Linear Algebraic Equations

In robotic analysis, there exist problems and situations, such as inverse kinematics, that we need to solve a set of coupled linear or nonlinear algebraic equations. Every numerical method of solving nonlinear equations also works by iteratively solving a set of linear equations.

Consider a system of $n$ linear algebraic equations with real constant coefficients,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots = \cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$ (9.1)

which can also be written in matrix form

$$[A] \mathbf{x} = \mathbf{b}.$$ (9.2)

There are numerous methods for solving this set of equations. Among the most efficient methods is the LU factorization method.

For every nonsingular matrix $[A]$ there exists an upper triangular matrix $[U]$ with nonzero diagonal elements and a lower triangular matrix $[L]$ with unit diagonal elements, such that

$$[A] = [L] [U]$$ (9.3)

$$[A] = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}$$ (9.4)

$$[L] = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
l_{21} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
l_{n1} & l_{n2} & \cdots & 1
\end{bmatrix}$$ (9.5)

The process of factoring $[A]$ into $[L][U]$ is called $LU$ factorization. Once the $[L]$ and $[U]$ matrices are obtained, the equation

$$[L][U]x = b$$  \hspace{1cm} (9.7)

can be solved by transforming into

$$[L]y = b$$  \hspace{1cm} (9.8)

and

$$[U]x = y.$$  \hspace{1cm} (9.9)

Equations (9.8) and (9.9) are both a triangular set of equations and their solutions are easy to obtain by forward and backward substitution.

**Proof.** To show how $[A]$ can be transformed into $[L][U]$, we consider a $4 \times 4$ matrix.

$$[A] = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
l_{21} & 1 & 0 & 0 \\
l_{31} & l_{32} & 1 & 0 \\
l_{41} & l_{42} & l_{43} & 1
\end{bmatrix} \begin{bmatrix}
u_{11} & u_{12} & u_{13} & u_{14} \\
u_{22} & u_{23} & u_{24} \\
u_{33} & u_{34} \\
u_{44}
\end{bmatrix}.$$  \hspace{1cm} (9.10)

Employing a dummy matrix $[B]$, we may combine the elements of $[L]$ and $[U]$ as

$$[B] = \begin{bmatrix}
u_{11} & u_{12} & u_{13} & u_{14} \\
l_{21} & u_{22} & u_{23} & u_{24} \\
l_{31} & l_{32} & u_{33} & u_{34} \\
l_{41} & l_{42} & l_{43} & u_{44}
\end{bmatrix}.$$  \hspace{1cm} (9.11)

The elements of $[B]$ will be calculated one by one, in the following order:

$$[B] = \begin{bmatrix}
(1) & (2) & (3) & (4) \\
(5) & (8) & (9) & (10) \\
(6) & (11) & (13) & (14) \\
(7) & (12) & (15) & (16)
\end{bmatrix}.$$  \hspace{1cm} (9.12)

The process for generating a matrix $[B]$, associated to an $n \times n$ matrix $[A]$, is performed in $n - 1$ iterations. After $i - 1$ iterations, the matrix is in the following form:

$$[B] = \begin{bmatrix}
u_{1,1} & u_{1,2} & \cdots & u_{1,i-1} & \cdots & u_{1,n} \\
l_{2,1} & u_{2,2} & \cdots & \cdots & \cdots & u_{2,n} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
l_{n,1} & l_{n,2} & \cdots & l_{n,i-1} & \cdots & u_{i-1,n}
\end{bmatrix}.$$  \hspace{1cm} (9.13)