Chapter 2
Discrete-Time Markov Models

2.1 Discrete-Time Markov Chains

Consider a system that is observed at times 0, 1, 2, . . . . Let $X_n$ be the state of the system at time $n$ for $n = 0, 1, 2, . . . . Suppose we are currently at time $n = 10$. That is, we have observed $X_0, X_1, \ldots, X_{10}$. The question is: can we predict, in a probabilistic way, the state of the system at time 11? In general, $X_{11}$ depends (in a possibly random fashion) on $X_0, X_1, \ldots, X_{10}$. Considerable simplification occurs if, given the complete history $X_0, X_1, \ldots, X_{10}$, the next state $X_{11}$ depends only upon $X_{10}$. That is, as far as predicting $X_{11}$ is concerned, the knowledge of $X_0, X_1, \ldots, X_9$ is redundant if $X_{10}$ is known. If the system has this property at all times $n$ (and not just at $n = 10$), it is said to have a Markov property. (This is in honor of Andrey Markov, who, in the 1900s, first studied the stochastic processes with this property.) We start with a formal definition below.

**Definition 2.1.** (Markov Chain) A stochastic process $\{X_n, n \geq 0\}$ on state space $S$ is said to be a discrete-time Markov chain (DTMC) if, for all $i$ and $j$ in $S$,

$$
P(X_{n+1} = j | X_n = i, X_{n-1}, \ldots, X_0) = P(X_{n+1} = j | X_n = i).
$$

(2.1)

A DTMC $\{X_n, n \geq 0\}$ is said to be time homogeneous if, for all $n = 0, 1, \ldots$,

$$
P(X_{n+1} = j | X_n = i) = P(X_1 = j | X_0 = i).
$$

(2.2)

Note that (2.1) implies that the conditional probability on the left-hand side is the same no matter what values $X_0, X_1, \ldots, X_{n-1}$ take. Sometimes this property is described in words as follows: given the present state of the system (namely $X_n$), the future state of the DTMC (namely $X_{n+1}$) is independent of its past (namely $X_0, X_1, \ldots, X_{n-1}$). The quantity $P(X_{n+1} = j | X_n = i)$ is called a one-step transition probability of the DTMC at time $n$. Equation (2.2) implies that, for time-homogeneous DTMCs, the one-step transition probability depends on $i$ and $j$ but is the same at all times $n$; hence the terminology time homogeneous.

In this chapter we shall consider only time-homogeneous DTMCs with finite state space $S = \{1, 2, \ldots, N\}$. We shall always mean time-homogeneous DTMC
when we say DTMC. For such DTMCs, we introduce a shorthand notation for the one-step transition probability:

\[ p_{i,j} = \mathbb{P}(X_{n+1} = j | X_n = i), \ i, j = 1, 2, \ldots, N. \]  

(2.3)

Note the absence of \( n \) in the notation. This is because the right-hand side is independent of \( n \) for time-homogeneous DTMCs. Note that there are \( N^2 \) one-step transition probabilities \( p_{i,j} \). It is convenient to arrange them in an \( N \times N \) matrix form as shown below:

\[
P = 
\begin{bmatrix}
p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,N} \\
p_{2,1} & p_{2,2} & p_{2,3} & \cdots & p_{2,N} \\
p_{3,1} & p_{3,2} & p_{3,3} & \cdots & p_{3,N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{N,1} & p_{N,2} & p_{N,3} & \cdots & p_{N,N}
\end{bmatrix}.
\]  

(2.4)

The matrix \( P \) in the equation above is called the one-step transition probability matrix, or transition matrix for short, of the DTMC. Note that the rows correspond to the starting state and the columns correspond to the ending state of a transition. Thus the probability of going from state 2 to state 3 in one step is stored in row number 2 and column number 3.

The information about the transition probabilities can also be represented in a graphical fashion by constructing a transition diagram of the DTMC. A transition diagram is a directed graph with \( N \) nodes, one node for each state of the DTMC. There is a directed arc going from node \( i \) to node \( j \) in the graph if \( p_{i,j} \) is positive; in this case, the value of \( p_{i,j} \) is written next to the arc for easy reference. We can use the transition diagram as a tool to visualize the dynamics of the DTMC as follows. Imagine a particle on a given node, say \( i \), at time \( n \). At time \( n+1 \), the particle moves to node 2 with probability \( p_{i,2} \), node 3 with probability \( p_{i,3} \), etc. \( X_n \) can then be thought of as the position (node index) of the particle at time \( n \).

**Example 2.1.** (Transition Matrix and Transition Diagram). Suppose \( \{X_n, n \geq 0\} \) is a DTMC with state space \( \{1, 2, 3\} \) and transition matrix

\[
P = 
\begin{bmatrix}
.20 & .30 & .50 \\
.10 & .00 & .90 \\
.55 & .00 & .45
\end{bmatrix}.
\]  

(2.5)

If the DTMC is in state 3 at time 17, what is the probability that it will be in state 1 at time 18? The required probability is \( p_{3,1} \) and is given by the element in the third row and the first column of the matrix \( P \). Hence the answer is .55.

If the DTMC is in state 2 at time 9, what is the probability that it will be in state 3 at time 10? The required probability can be read from the element in the second row and third column of \( P \). It is \( p_{2,3} = .90 \).

The transition diagram for this DTMC is shown in Figure 2.1. Note that it has no arc from node 3 to node 2, representing the fact that \( p_{3,2} = 0 \).