3

Continued Fraction Expansions

3.1 Introduction

A continued fraction \([2, 3, 4, 5]\) is a sequence of fractions

\[
    f_n = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots + \frac{a_n}{b_n}}}}
\]  

(3.1)

formed from two sequences \(a_1, a_2, \ldots\) and \(b_0, b_1, \ldots\) of numbers. For typographical convenience, definition (3.1) is usually recast as

\[
    f_n = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots + \frac{a_n}{b_n}}}}
\]

In many practical examples, the approximant \(f_n\) converges to a limit, which is typically written as

\[
    \lim_{n \to \infty} f_n = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots}}}
\]

Because the elements \(a_n\) and \(b_n\) of the two defining sequences can depend on a variable \(x\), continued fractions offer an alternative to power series in expanding functions such as distribution functions. In fact, continued fractions can converge where power series diverge, and where both types of expansions converge, continued fractions often converge faster.

A lovely little example of a continued fraction is furnished by

\[
    \sqrt{2} - 1 = \frac{1}{2 + (\sqrt{2} - 1)}
\]

\[
    = \frac{1}{2 + \frac{1}{2 + (\sqrt{2} - 1)}}
\]

\[
    = \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + (\sqrt{2} - 1)}}}
\]
One can easily check numerically that the limit
\[ \sqrt{2} = 1 + \frac{1}{2+ \frac{1}{2+ \frac{1}{2+ \cdots}}} \]
is correct. It is harder to prove this analytically. For the sake of brevity, we will largely avoid questions of convergence. Readers interested in a full treatment of continued fractions can consult the references [2, 3, 5]. Problems 7 through 10 prove convergence when the sequences \(a_n\) and \(b_n\) are positive.

Before giving more examples, it is helpful to consider how we might go about evaluating the approximant \(f_n\). One obvious possibility is to work from the bottom of the continued fraction (3.1) to the top. This obvious approach can be formalized by defining fractional linear transformations \(t_0(x) = b_0 + x\) and \(t_n(x) = a_n/(b_n + x)\) for \(n > 0\). If the circle symbol \(\circ\) denotes functional composition, then we take \(x = 0\) and compute
\[
\begin{align*}
t_n(0) &= \frac{a_n}{b_n}, \\
t_{n-1} \circ t_n(0) &= \frac{a_{n-1}}{b_{n-1} + t_n(0)}, \\
&\vdots \\
t_0 \circ t_1 \circ \cdots \circ t_n(0) &= f_n.
\end{align*}
\]
This turns out to be a rather inflexible way to proceed because if we want the next approximant \(f_{n+1}\), we are forced to start all over again. In 1655 J. Wallis [6] suggested an alternative strategy. (This is a venerable but often neglected subject.)

### 3.2 Wallis’s Algorithm

According to Wallis,
\[
t_0 \circ t_1 \circ \cdots \circ t_n(x) = \frac{A_{n-1}x + A_n}{B_{n-1}x + B_n} \quad (3.2)
\]
for a certain pair of auxiliary sequences \(A_n\) and \(B_n\). Taking \(x = 0\) gives the approximant \(f_n = A_n/B_n\). The sequences \(A_n\) and \(B_n\) satisfy the initial conditions
\[
\begin{pmatrix} A_{-1} \\ B_{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} b_0 \\ 1 \end{pmatrix} \quad (3.3)
\]
and for \(n > 0\) the recurrence relation
\[
\begin{pmatrix} A_n \\ B_n \end{pmatrix} = b_n \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix} + a_n \begin{pmatrix} A_{n-2} \\ B_{n-2} \end{pmatrix}. \quad (3.4)
\]