THE RADON TRANSFORM ON TWO-POINT HOMOGENEOUS SPACES

Let $X$ be a complete Riemannian manifold, $x$ a point in $X$ and $X_x$ the tangent space to $X$ at $x$. Let $\text{Exp}_x$ denote the mapping of $X_x$ into $X$ given by $\text{Exp}_x(u) = \gamma_u(1)$, where $t \to \gamma_u(t)$ is the geodesic in $X$ through $x$ with tangent vector $u$ at $x = \gamma_u(0)$.

A connected submanifold $S$ of a Riemannian manifold $X$ is said to be **totally geodesic** if each geodesic in $X$ which is tangential to $S$ at a point lies entirely in $S$.

The totally geodesic submanifolds of $\mathbb{R}^n$ are the planes in $\mathbb{R}^n$. Therefore, in generalizing the Radon transform to Riemannian manifolds, it is natural to consider integration over totally geodesic submanifolds. In order to have enough totally geodesic submanifolds at our disposal we consider in this section Riemannian manifolds $X$ which are two-point homogeneous in the sense that for any two-point pairs $p, q \in X, p', q' \in X$, satisfying $d(p, q) = d(p', q')$, (where $d =$ distance), there exists an isometry $g$ of $X$ such that $g \cdot p = p', g \cdot q = q'$. We start with the subclass of Riemannian manifolds with the richest supply of totally geodesic submanifolds, namely the spaces of constant curvature.

While §1, which constitutes most of this chapter, is elementary, §§2–§4 will involve a bit of Lie group theory.

§1 Spaces of Constant Curvature. Inversion and Support Theorems

Let $X$ be a simply connected complete Riemannian manifold of dimension $n \geq 2$ and constant sectional curvature.

**Lemma 1.1.** Let $x \in X$, $V$ a subspace of the tangent space $X_x$. Then $\text{Exp}_x(V)$ is a totally geodesic submanifold of $X$.

**Proof.** For this we choose a specific embedding of $X$ into $\mathbb{R}^{n+1}$, and assume for simplicity the curvature is $\epsilon(= \pm 1)$. Consider the quadratic form

$$B_\epsilon(x) = x_1^2 + \cdots + x_n^2 + \epsilon x_{n+1}^2$$

and the quadric $Q_\epsilon$ given by $B_\epsilon(x) = \epsilon$. The orthogonal group $O(B_\epsilon)$ acts transitively on $Q_\epsilon$. The form $B_\epsilon$ is positive definite on the tangent space $\mathbb{R}^n \times (0)$ to $Q_\epsilon$ at $x^0 = (0, \ldots, 0, 1)$; by the transitivity $B_\epsilon$ induces a positive definite quadratic form at each point of $Q_\epsilon$, turning $Q_\epsilon$ into a
Riemannian manifold, on which $O(B_\epsilon)$ acts as a transitive group of isometries. The isotropy subgroup at the point $x^0$ is isomorphic to $O(n)$ and its acts transitively on the set of 2-dimensional subspaces of the tangent space $(Q_\epsilon)x^0$. It follows that all sectional curvatures at $x^0$ are the same, namely $\epsilon$, so by homogeneity, $Q_\epsilon$ has constant curvature $\epsilon$. In order to work with connected manifolds, we replace $Q_{-1}$ by its intersection $Q_{-1}^+$ with the half-space $x_{n+1} > 0$. Then $Q_{+1}$ and $Q_{+1}^+$ are simply connected complete Riemannian manifolds of constant curvature. Since such manifolds are uniquely determined by the dimension and the curvature it follows that we can identify $X$ with $Q_{+1}$ or $Q_{+1}^+$.

The geodesic in $X$ through $x^0$ with tangent vector $(1, 0, \ldots, 0)$ will be left point-wise fixed by the isometry

$$(x_1, x_2, \ldots, x_n, x_{n+1}) \rightarrow (x_1, -x_2, \ldots, -x_n, x_{n+1}).$$

This geodesic is therefore the intersection of $X$ with the two-plane $x_2 = \cdots = x_n = 0$ in $\mathbb{R}^{n+1}$. By the transitivity of $O(n)$ all geodesics in $X$ through $x^0$ are intersections of $X$ with two-planes through $0$. By the transitivity of $O(Q_\epsilon)$ it then follows that the geodesics in $X$ are precisely the nonempty intersections of $X$ with two-planes through the origin.

Now if $V \subset X_{x^0}$ is a subspace, $\text{Exp}_{x^0}(V)$ is by the above the intersection of $X$ with the subspace of $\mathbb{R}^{n+1}$ spanned by $V$ and $x^0$. Thus $\text{Exp}_{x^0}(V)$ is a quadric in $V + \mathbb{R}x^0$ and its Riemannian structure induced by $X$ is the same as induced by the restriction $B_\epsilon|(V + \mathbb{R}x^0)$. Thus, by the above, the geodesics in $\text{Exp}_{x^0}(V)$ are obtained by intersecting it with two-planes in $V + \mathbb{R}x^0$ through $0$. Consequently, the geodesics in $\text{Exp}_{x^0}(V)$ are geodesics in $X$ so $\text{Exp}_{x^0}(V)$ is a totally geodesic submanifold of $X$. By the homogeneity of $X$ this holds with $x^0$ replaced by an arbitrary point $x \in X$. The lemma is proved.

It will be convenient in the following to use another model of the space $X = Q_\epsilon^+$, namely the generalization of the space $H^2$ in Ch. II, Theorem 4.2. It is the hyperbolic space $H^n$ which is the unit ball $|y| < 1$ in $\mathbb{R}^n$ with the Riemannian structure $ds^2$ related to the flat $ds_0^2$ by

$$ds^2 = \frac{4(dy_1^2 + \cdots + dy_n^2)}{(1 - y_1^2 - \cdots - y_n^2)^2} = \rho^2 ds_0^2.$$

Consider the mapping $y = \Phi(x)$, $x \in Q_\epsilon^+$, given by

$$y_1, \ldots, y_n = \frac{1}{x_{n+1} + 1}(x_1, \ldots, x_n),$$

with inverse

$$(x_1, \ldots, x_n) = \frac{2}{1 - |y|^2}(y_1, \ldots, y_n), \quad x_{n+1} = \frac{1 + |y|^2}{1 - |y|^2}.$$
