We know from standard normal linear systems theory that controllability and observability are a pair of very important concepts for linear systems. They provide us with certain structural features of the system as well as some internal properties of the system, and are essential for certain system design problems studied in the later chapters.

This chapter focuses on topics related with controllability and observability. As a preliminary, the first section introduces the concept of reachable state and characterizes the state reachable subspaces. In Sects. 4.2 and 4.3, definitions of controllability and observability are proposed, respectively. As we will find out, unlike the normal linear system case, there are several types of controllabilities and observabilities for descriptor linear systems. Conditions for these types of controllabilities and observabilities in terms of the fast and slow subsystems are given. Based on the results introduced in Sects. 4.2 and 4.3, the so-called dual principle is proposed in Sect. 4.4, which reveals the close relation between the controllabilities and observabilities of a regular descriptor linear system and its dual system. This principle gives us great convenience in the sequential sections in deriving criteria for various types of controllability and observability. Sections 4.5 and 4.6 further give some criteria for these types of controllabilities and observabilities of descriptor systems. Section 4.5 stresses on those using the original system coefficient matrices, while Sect. 4.6 gives criteria based on certain restricted equivalent forms. Afterwards, two problems which are closely related with the concepts of controllability and observability, namely, system structural decomposition and minimal realization, are investigated in Sects. 4.7 and 4.8. Notes and references follow in Sect. 4.9.

4.1 State Reachable Subsets

Consider the regular descriptor system:

\[ E \dot{x} = Ax + Bu, \]  
(4.1)
where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^r \) are the state vector and the control input vector, respectively; and \( E, A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times r} \) are constant matrices. The dynamical order of the system is \( n_0 = \text{rank} E \leq n \).

It follows from the standard decomposition of descriptor linear systems that there exist two nonsingular matrices \( Q \) and \( P \) such that under the transformation \( (P, Q) \) the system (4.1) is transformed into the following equivalent standard decomposition form:

\[
\begin{aligned}
\dot{x}_1 &= A_1 x_1 + B_1 u \\
N \dot{x}_2 &= x_2 + B_2 u,
\end{aligned}
\]

where \( x_1 \in \mathbb{R}^{n_1} \), \( x_2 \in \mathbb{R}^{n_2} \), \( n_1 + n_2 = n \), the matrix \( N \in \mathbb{R}^{n_2 \times n_2} \) is nilpotent, and the nilpotent index is denoted by \( h \). The relations between the states and the coefficients of the two equivalent systems are given below:

\[
\begin{aligned}
P^{-1}x &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\
QEP &= \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \\
QAP &= \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}, \\
QB &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.
\end{aligned}
\]

### 4.1.1 The Definition

For the sake of simplicity, in this section we discuss the descriptor linear system in the standard decomposition form (4.2). The obtained results are easily applicable to systems in the general form of (4.1).

Let us start from the concept of the reachable states.

**Definition 4.1.** Given the regular descriptor linear system (4.2), a vector \( w \in \mathbb{R}^n \) is said to be a reachable vector of system (4.2), if there exists an initial condition \( x_1(0) = x_{10} \), an admissible control input \( u(t) \in \mathcal{C}_p^{h-1} \), and some \( t_1 > 0 \) such that the response of the system (4.2) satisfies

\[
\begin{bmatrix} x_1(t_1) \\ x_2(t_1) \end{bmatrix} = w.
\]

Please note that in the above definition the admissible control input is confined to \( u(t) \in \mathcal{C}_p^{h-1} \), which represents the set of \( h - 1 \) times piecewise continuously differentiable functions.

Let

\[
\mathcal{R}_t [x_{10}] = \{ w | \exists u(t) \in \mathcal{C}_p^{h-1} \text{ s.t. } x(t, u, x_{10}) = w \in \mathbb{R}^n \},
\]

then \( \mathcal{R}_t [x_{10}] \) is the set of reachable states at time \( t \) from the initial condition \( x_1(0) = x_{10} \). It clearly reduces to \( \mathcal{R}_t [0] \) when \( x_{10} = 0 \). Note that the response of the system (4.2) is given by