CHAPTER 10

Description of Internal Deformation
and Forces

10.1 Introduction

Since living organs normally go through finite deformation, a bioengineer should know the subject of finite deformation analysis. This subject is not difficult, but it usually lies outside the common engineering curriculum. It is not simple, and considerable patience is needed to master it. In the following, a presentation of its most important aspects is given in easy to understand physical terms. There are many books and papers on this subject (see References). Fung (1965) is believed to be one of the easiest to read.

The concept of strain is reviewed in Secs. 10.2 and 10.3. The concept of stress is discussed in Sec. 10.4. The equation of motion is presented in Sec. 10.5, work and energy in Sec. 10.6, the use of the strain energy function in Sec. 10.7, and complementary energy function in Sec. 10.8, all without linearization. The separation of local rotational motion and strain in a general deformation of a continuum is discussed in Sec. 10.9.

Of these topics, the most important are the definitions of Cauchy's, Lagrange's, and Kirchhoff's stress tensors discussed in Sec. 10.4. They are denoted by $\sigma_{ij}$, $T_{ij}$ and $S_{ij}$, respectively. The stresses $\sigma_{ij}$, $S_{ij}$ are symmetric tensors, but $T_{ij}$ is not. Then it is shown in Sec. 10.7 that if the material is elastic and the strain energy function per unit initial volume, $\rho_0 W$, is expressed in terms of the Green's strain tensor, $E_{ij}$, we have

$$S_{ij} = \frac{\partial(\rho_0 W)}{\partial E_{ij}}.$$  \hspace{1cm} (1)

On the other hand, if $\rho_0 W$ is expressed in terms of the deformation gradient tensor $\partial x_i/\partial a_j$, then
If the strain energy per unit initial volume is expressed as a function of the Kirchhoff stress tensor \(S_{ij}\), then it is called the complementary strain energy denoted by \(\rho_0 W_c\). We have

\[
E_{ij} = \frac{\partial \rho_0 W_c}{\partial S_{ij}}.
\]

These important formulas are used frequently in biomechanics.

After these mathematical concepts are clarified, we return to biology. The first question we ask is whether living tissues are elastic. The answer, unfortunately, if no. But the concepts of pseudo-elasticity and the pseudo-strain energy function provides us with a useful approximation. Applications of these concepts are discussed in Sec. 10.8 and continued in Chap. 11.

### 10.2 Description of Internal Deformation

To describe the deformation of a body we need to know the position of any point in the body with respect to an initial configuration which shall be called the reference state. Moreover, to describe position we need a frame of reference. Let us choose a rectangular cartesian frame of reference, Fig. 10.2: 1. Every material particle in the body at the reference state \(S_0\) has three coordinates \(a_1, a_2, a_3\) which can be written as a column matrix denoted by any one of the following forms:

\[
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}
\quad \text{or} \quad \{a_1, a_2, a_3\} \quad \text{or} \quad \{a_i\} \quad \text{or} \quad a_i, \quad i = 1, 2, 3.
\]

A particle \(P\) has the coordinates \(a_i\), and a neighboring particle \(P'\) has the coordinates \(a_i + da_i\). When the body is deformed the particles \(P, P'\) are moved to \(Q, Q'\) whose coordinates are \(x_i\) and \(x_i + dx'_i\), respectively. The deformation of the body is known completely if we know the relationship

\[
x_i = x_i(a_1, a_2, a_3) \quad i = 1, 2, 3 \tag{1a}
\]

or its inverse

\[
a_i = a_i(x_1, x_2, x_3) \quad i = 1, 2, 3 \tag{1b}
\]

for every point in the body. If we write

\[
x_i = a_i + u_i \quad i = 1, 2, 3 \tag{1c}
\]

then \(u_i\) is called the displacement of the particle \(P\).

In general, we do not have the luxury of knowing the transformation law (or mapping function), as expressed in Eq. (1a) or (1b), at the beginning of