Chapter 5
The hypergeometric differential equation

Differential equations with three regular singular points (one located at infinity) play an important role in mathematical physics, and in this chapter we investigate this situation in some detail. More details are found in the rich literature on the subject, see, e.g., Refs. 11, 18, 31.

5.1 Basic properties

We have already seen that the hypergeometric differential equation in (4.3)

\[ z(z-1)u''(z) + \left[ (\alpha + \beta + 1)z - \gamma \right] u'(z) + \alpha \beta u(z) = 0 \]

has three regular singular points at \( z = 0, 1, \infty \). We also write the differential equation as

\[ u''(z) + \frac{(\alpha + \beta + 1)z - \gamma}{z(z-1)} u'(z) + \frac{\alpha \beta}{z(z-1)} u(z) = 0 \]

The roots of the indicial equation are given by, see (2.12),

\[ I(\lambda) = \lambda^2 + (p_0 - 1) \lambda + q_0 = 0 \]

The roots of the indicial equations at the three different regular singular points are summarized in Table 5.1.

Table 5.1 The roots of the indicial equation of the hypergeometric differential equation.

<table>
<thead>
<tr>
<th>Point</th>
<th>( p_0 )</th>
<th>( q_0 )</th>
<th>Roots ( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z = 0 )</td>
<td>( \gamma )</td>
<td>0</td>
<td>0, 1 - ( \gamma )</td>
</tr>
<tr>
<td>( z = 1 )</td>
<td>( 1 + \alpha + \beta - \gamma )</td>
<td>0</td>
<td>0, ( \gamma - \alpha - \beta )</td>
</tr>
<tr>
<td>( z = \infty )</td>
<td>( 1 - \alpha - \beta )</td>
<td>( \alpha \beta )</td>
<td>( \alpha, \beta )</td>
</tr>
</tbody>
</table>
We now investigate the solution of the hypergeometric differential equation in the vicinity of the regular singular point \( z = 0 \), where the roots of the indicial equation are 0 and \( 1 - \gamma \). Therefore, we expect that one of the solutions is analytic at \( z = 0 \). In general, from the analysis in Section 2.4, there are two linearly independent solutions \( u(z) \) and \( v(z) \) of the form

\[
\begin{align*}
    u(z) &= 1 + \sum_{n=1}^{\infty} a_n z^n, \\
    v(z) &= z^{1-\gamma} \left( 1 + \sum_{n=1}^{\infty} a'_n z^n \right)
\end{align*}
\]  

(5.1)

Note that the solution \( v(z) \) is not analytic near the regular singular point \( z = 0 \), unless \( \gamma = 1,0,-1,-2,\ldots \). However, in this case the two indicial roots differ by an integer, and \( \operatorname{Re}(1 - \gamma) \geq 0 \), which implies that the results of Section 2.4.3 have to be used. The power series solution \( u(z) \) in (5.1) is called the **hypergeometric series**. The radius of convergence of this power series is at most the distance to the closest singular point,\(^1\) i.e., the radius is less than or equal to 1. The properties of this power series solution are presented in Theorem 5.1 below.

The analytic extension of the hypergeometric series in the complex \( z \)-plane is denoted the **hypergeometric function**,\(^2\) and we adopt the notation \( F(\alpha, \beta; \gamma; z) \). Note that

\[
F(\alpha, \beta; \gamma; 0) = 1
\]

The hypergeometric function is analytic everywhere in the finite complex plane, excluding the possible singular point at \( z = 1 \). The singularity can be either a pole or a branch point. If the singularity is a branch point, we introduce a branch cut from \( z = 1 \) to \( z = \infty \) along the real axis, in order to get a one-valued function throughout the cut plane.

A long list of elementary and special functions can be expressed in a direct or an indirect way in the hypergeometric function. A collection of functions that can be expressed in the hypergeometric function is found in Appendix E on page 209.

### 5.2 Hypergeometric series

The objective of our investigation is now to explicitly determine the coefficients \( a_n \) and \( a'_n \) in (5.1). This approach resembles strongly the analysis in Section 2.4, i.e., Frobenius method, but due to the special form of the differential equation, we repeat parts of the analysis here.

We begin with the \( a_n \) coefficients and insert the power series of \( u(z) \) in the differential equation, and identity the coefficient in front of the same power of \( z \), which must vanish in order to have the power series of \( u(z) \) satisfying the differential equa-

\(^1\) An exception occurs if \( \alpha \beta = 0 \), then \( u(z) = 1 \) is a solution, which has infinite convergence radius.

\(^2\) The more complete notion is \( _2F_1(\alpha, \beta; \gamma; z) \), see also the generalized hypergeometric series in Equation (7.14) on page 138.