Maximal Inequalities as Necessary Conditions for Almost Everywhere Convergence*

By

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Introduction

Consider the inequality
\[ \mu(T^*f > \lambda) \leq K \int_{\Omega} |f|^p d\mu / \lambda^p. \]

Here \((\Omega, \mathcal{A}, \mu)\) is a \(\sigma\)-finite positive measure space, \(1 \leq p < \infty, f \in L_p(\Omega, \mathcal{A}, \mu),\)
\(T^*f(\omega) = \sup_{1 \leq n < \infty} |T_n f(\omega)|\) where each \(T_n\) is a bounded linear operator in \(L_p, \lambda > 0,\)
and \(K\) is a real number not depending on \(f\) or \(\lambda.\)

Such an inequality, or one similar, is often the central part of proofs establishing the almost everywhere convergence of \(\{T_n f\}\) for every \(f\) in \(L_p.\) The inequality implies, by letting \(\lambda \to \infty,\) that \(T^*f < \infty\) almost everywhere for all \(f\) in \(L_p.\) Thus, by the Banach convergence theorem (see, for example, page 332 of [6]), the almost everywhere convergence of \(\{T_n f\}\) for every \(f\) in \(L_p\) follows from the almost everywhere convergence of \(\{T_n f\}\) for \(f\) in a dense subset of \(L_p.\) Convergence in a dense subset is often rather easy to establish. It is not uncommon, however, that proving the above sort of inequality, the maximal inequality, is genuinely difficult.

Can one know in advance whether the maximal inequality approach to proving almost everywhere convergence is plausible? Conceivably, almost everywhere convergence could hold for the particular problem in which one is interested without a maximal inequality of the above form holding. When is such an inequality a necessary condition for almost everywhere convergence?

A. P. Calderón ([12], II, page 165) and E. M. Stein [10] have obtained important results in this direction for operators arising in Fourier analysis. For Stein, who generalizes Calderón's result considerably, \(\Omega\) is the homogeneous space of a compact group and each \(T_n\) commutes with translations. However, many convergence problems in analysis, for example, those most often encountered in probability theory and ergodic theory, do not have this kind of setting.

In Section 1 of this paper we pose the necessity question somewhat more generally for arbitrary sequences of measurable functions. We consider a set \(\mathcal{C}\) of such sequences and show that if each sequence in \(\mathcal{C}\) converges almost everywhere (actually less is needed) and \(\mathcal{C}\) satisfies one other condition, then \(\mathcal{C}\) satisfies a maximal inequality (Theorems 1 and 2). These results apply to many problems in ergodic theory, probability theory, orthogonal series, and the like. Some of

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these applications are discussed in the succeeding sections. The key requirement on \( \mathcal{C} \) in Theorem 1 is that \( \mathcal{C} \) be stochastically convex. This condition, defined in Section 1, is usually quite easy to check in the applications and implies that \( \mathcal{C} \) is not too small.

Throughout the paper the same symbol is used for a measurable function and for the equivalence class (of measurable functions any two of which are equal almost everywhere) containing it. Expressions involving equality and inequality signs are sometimes to be interpreted as holding almost everywhere. Also, \( \int f \) is occasionally used to denote the integral over \( \Omega \) of \( f \) relative to the measure \( \mu \).

1. A basic question

Let \( (\Omega, \mathcal{A}, \mu) \) be a positive measure space with \( \mu(\Omega) = 1 \). Let \( \mathcal{D} \) be the collection of all sequences \( f = (f_1, f_2, \ldots) \) with each \( f_n \) an \( \mathcal{A} \)-measurable function from \( \Omega \) into the complex numbers. For \( f = (f_1, f_2, \ldots) \) in \( \mathcal{D} \) define \( f^* \) by \( f^*(\omega) = \sup_{1 \leq n < \infty} |f_n(\omega)|, \omega \in \Omega \). Let \( \mathcal{C} \subset \mathcal{D} \) and \( 0 < p < \infty \).

**Question.** What conditions on \( \mathcal{C} \) assure the existence of a real number \( K \) satisfying

\[
\mu(f^* > \lambda) \leq K/\lambda^p, \quad \lambda > 0, \quad f \in \mathcal{C}.
\]

In particular, under what conditions on \( \mathcal{C} \) does the almost everywhere convergence of each sequence in \( \mathcal{C} \) imply the existence of such a \( K \)?

Note that the right hand side of (1) does not depend on \( f \). This usually causes no difficulty in the applications and can often be accomplished by demanding that if \( f = (f_1, f_2, \ldots) \in \mathcal{C} \), then \( \int |f_1|^p \leq 1 \) or some other similar condition. Also, the condition \( \mu(\Omega) = 1 \) can often be dropped in the applications.

If \( f \) and \( g \) belong to \( \mathcal{D} \) write \( f \sim g \) if \( f \) and \( g \) have the same distribution, that is, if \( \int \phi(f) = \int \phi(g) \) for all bounded Baire functions \( \phi \) on the obvious product space. Clearly, if \( f \) and \( g \) belong to \( \mathcal{D} \) and \( f \sim g \), then \( f^* \sim g^* \) where the notation is to denote again that \( f^* \) and \( g^* \) have the same distribution.

We shall say that \( \mathcal{C} \) is stochastically convex if the following condition is satisfied: Each term of each sequence in \( \mathcal{C} \) is nonnegative almost everywhere, and if \( f_k = (f_{k1}, f_{k2}, \ldots) \in \mathcal{C} \), \( k = 1, 2, \ldots \), then there are sequences \( g_k = (g_{k1}, g_{k2}, \ldots) \in \mathcal{D} \), \( k = 1, 2, \ldots \), such that

(i) the \( g_k \)'s are (stochastically) independent,

(ii) \( f_k \sim g_k, \quad k = 1, 2, \ldots \),

(iii) if \( \{a_k\} \) is a nonnegative number sequence with \( \sum_{k=1}^{\infty} a_k = 1 \), then there is an \( h \in \mathcal{C} \) such that \( h \sim \sum_{k=1}^{\infty} a_k g_{kn} \).

Note that (i) is equivalent to saying that the rows of the matrix \( (g_{kn}) \) are independent.

The finiteness condition is satisfied if \( \mu(f^* < \infty) > 0, \ f \in \mathcal{C} \). Clearly, if each \( f \in \mathcal{C} \) converges almost everywhere to a finite limit then the finiteness condition is satisfied.