Chapter 4
The Mass Matrix

While the mass and rotational inertia of a body are the elemental quantities for studying single rigid body dynamics, the mass matrix plays the corresponding role for multibody systems. One crucial difference is that the mass matrix varies with the configuration of the system. In this chapter, we derive expressions for the mass matrix of a serial-chain multibody system, study its properties, and develop related computational algorithms.

4.1 Mass Matrix of a Serial-Chain System

4.1.1 Kinetic Energy of the Serial-Chain

For a serial-chain system, we use (2.7) to define the $M(k)$ spatial inertia of the $k$th link about its body frame $B_k$ as follows:

$$M(k) \triangleq \begin{pmatrix} J(k) & m(k)\vec{p}(k) \\ -m(k)\vec{p}(k) & m(k)I \end{pmatrix}$$

(4.1)

$J(k)$ is the inertia tensor for the $k$th link about $B_k$, $p(k)$ is the vector from $B_k$ to the center of mass of the $k$th link, and $m(k)$ is the mass of the $k$th link.

The total kinetic energy of the system, $\mathcal{K}_e$, is the sum of the kinetic energies of each of the individual links:

$$\mathcal{K}_e \triangleq \frac{1}{2} \sum_{k=1}^{n} \mathcal{V}^*(k)M(k)\mathcal{V}(k)$$

(4.2)
Equation (4.2) can be re-expressed as

\[
\mathcal{K}_e = \frac{1}{2} \left[ \mathcal{V}^*(1), \cdots, \mathcal{V}^*(n) \right] \begin{pmatrix}
M(1) & 0 & \cdots & 0 \\
0 & M(2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & M(n)
\end{pmatrix} \begin{pmatrix}
\mathcal{V}(1) \\
\mathcal{V}(2) \\
\vdots \\
\mathcal{V}(n)
\end{pmatrix}
\]

\[= \frac{1}{2} \mathcal{V}^* M \mathcal{V} \tag{4.3}\]

where \(M\) is the block-diagonal, symmetric and positive semi-definite spatial inertia spatial operator defined as

\[
M \triangleq \text{diag}\left\{ M[k] \right\}_{k=1}^n \in \mathbb{R}^{6n \times 6n} \tag{4.4}\]

Equation (4.3) expresses the total system kinetic energy in terms of the spatial velocity vector \(\mathcal{V}\). Substituting \(\mathcal{V} = \phi^* H^* \dot{\theta}\) from (3.52) into (4.3) allows us to express the kinetic energy of the serial-chain in terms of the generalized velocities \(\dot{\theta}\):

\[
\mathcal{K}_e = \frac{1}{2} \phi^* H M \phi^* H^* \dot{\theta} = \frac{1}{2} \dot{\theta}^* M(\theta) \dot{\theta} \tag{4.5}\]

where

\[
M(\theta) \triangleq H \phi M \phi^* H^* \in \mathbb{R}^{N \times N} \tag{4.6}\]

Equation (4.5) has the familiar quadratic form for kinetic energy. \(M\) in (4.6) is referred to as the mass matrix of the multibody system. Since \(M\) is symmetric and positive semi-definite, so is the \(M\) mass matrix. However, since \(\phi\) is configuration-dependent, the mass matrix is also configuration-dependent. The mass matrix is a natural generalization of the notion of spatial inertia of single rigid bodies for articulated multibody systems and plays a central role in their dynamics.

Equation (4.6) can also be viewed as a spatial operator factorization of the mass matrix. This factored form of the mass matrix is referred to as the Newton–Euler Operator Factorization of the mass matrix. We will further explore the rich internal structure of the mass matrix in Chap. 7, and develop an alternative analytical Innovations Operator Factorization for it, as well as an expression for its inverse.

We now turn to the problem of computing the configuration dependent mass matrix. Possible methods include:

1. The Lagrangian approach which consists of forming a configuration dependent expression for the kinetic energy in (4.2), and taking its second order partial derivative with respect to the generalized velocities to obtain the mass matrix elements. This approach can become very complex for even moderately sized systems.

2. An approach based on (4.6). Here, the component operators \(\phi\), \(M\), and \(H\) are computed explicitly, and the mass matrix is computed by evaluating the prod-