Chapter 7
Mass Matrix Inversion and AB Forward Dynamics

This chapter uses the articulated body model to derive a new Innovations Operator Factorization of the serial-chain mass matrix, $\mathcal{M}$. This factorization has square factors, in contrast to the non-square Newton–Euler factorization of $\mathcal{M} = \mathcal{H}\phi\mathcal{M}\phi^*\mathcal{H}^*$ in (5.23). The Innovations factorization is subsequently used to obtain an explicit analytical operator expression for the inverse of the mass matrix. These factorizations have many important uses, which will be explored in subsequent chapters.

At this stage, we continue to focus on serial-chain rigid multibody systems. However, we will see later that these operator factorization and inversion results and identities continue to hold for a very broad class of multibody systems.

7.1 Articulated Body Spatial Operators

First, we summarize the recursive relationships for articulated body quantities defined in Algorithm 6.1 on page 106 in the previous chapter:

\[
\begin{align*}
\mathcal{P}^+(0) &= 0, \quad \tau(0) = 0 \\
\text{for } k = 1 \cdots n \\
\psi(k, k-1) &= \phi(k, k-1)\tau(k-1) \\
\mathcal{P}(k) &= \phi(k, k-1)\mathcal{P}^+(k-1)\phi^*(k, k-1) + \mathcal{M}(k) \\
\mathcal{D}(k) &= \mathcal{H}(k)\mathcal{P}(k)\mathcal{H}^*(k) \\
\mathcal{G}(k) &= \mathcal{P}(k)\mathcal{H}^*(k)\mathcal{D}^{-1}(k) \\
\mathcal{K}(k+1, k) &= \phi(k+1, k)\mathcal{G}(k) \\
\tau(k) &= \mathcal{I} - \mathcal{G}(k)\mathcal{H}(k) \\
\mathcal{P}^+(k) &= \tau(k)\mathcal{P}(k)
\end{align*}
\] (7.1)
$\psi(k+1,k)$ is the force/velocity transformation matrix for the articulated body model analogous to the $\phi(k+1,k)$ transformation matrix for the composite body model. While $\phi(k+1,k)$ is a function of kinematical quantities alone, $\psi(k+1,k)$ depends additionally on the hinge types and spatial inertias of all the outboard links. Also, unlike $\phi(k+1,k)$, $\psi(k+1,k)$ is not invertible.

Now define the block diagonal operator $P$ as

$$P \triangleq \text{diag} \left\{ P(k) \right\}_{k=1}^{n} \in \mathbb{R}^{6n \times 6n} \quad (7.2)$$

and the following additional spatial operators:

$$D \triangleq \text{diag} \left\{ D(k) \right\}_{k=1}^{n} = HPH^* \quad \in \mathbb{R}^{N \times N}$$

$$G \triangleq \text{diag} \left\{ G(k) \right\}_{k=1}^{n} = PH^*D^{-1} \quad \in \mathbb{R}^{6n \times N}$$

$$K \triangleq E_\psi G \quad \in \mathbb{R}^{6n \times N}$$

$$\tau \triangleq \text{diag} \left\{ \tau(k) \right\}_{k=1}^{n} = GH \quad \in \mathbb{R}^{6n \times 6n} \quad (7.3)$$

$$\overline{\tau} \triangleq \text{diag} \left\{ \overline{\tau}(k) \right\}_{k=1}^{n} = I - \tau \quad \in \mathbb{R}^{6n \times 6n}$$

$$\mathcal{P}^+ \triangleq \text{diag} \left\{ \mathcal{P}^+(k) \right\}_{k=1}^{n} = \overline{\tau}P\tau^* = \overline{\tau}P = \overline{\tau}^* \quad \in \mathbb{R}^{6n \times 6n}$$

$$E_\psi \triangleq E_\phi \overline{\tau} \quad \in \mathbb{R}^{6n \times 6n}$$

The operators $D, G, \tau, \overline{\tau}$, and $\mathcal{P}^+$ are all block diagonal, with the $k$th diagonal entry being defined by the recursions in (7.1). On the other hand, $K$ has non-zero elements only along its first sub-diagonal and has the form:

$$K = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
\mathcal{K}(2,1) & 0 & \ldots & 0 & 0 \\
0 & \mathcal{K}(3,2) & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \mathcal{K}(n, n-1) & 0
\end{pmatrix} \quad (7.4)$$

$E_\psi$ has the following similar structure:

$$E_\psi = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
\psi(2,1) & 0 & \ldots & 0 & 0 \\
0 & \psi(3,2) & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \psi(n, n-1) & 0
\end{pmatrix} \quad (7.5)$$