Chapter 7
Boundary Points and Resolution

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Abstract We use the notion of boundary points to study resolution proofs. Given a CNF formula $F$, an $l(x)$-boundary point is a complete assignment falsifying only clauses of $F$ having the same literal $l(x)$ of variable $x$. An $l(x)$-boundary point $p$ mandates a resolution on variable $x$. Adding the resolvent of this resolution to $F$ eliminates $p$ as an $l(x)$-boundary point. Any resolution proof has to eventually eliminate all boundary points of $F$. Hence one can study resolution proofs from the viewpoint of boundary point elimination. We use equivalence checking formulas to compare proofs of their unsatisfiability built by a conflict-driven SAT-solver and very short proofs tailored to these formulas. We show experimentally that in contrast to proofs generated by this SAT-solver, almost every resolution of a specialized proof eliminates a boundary point. This implies that one may use the share of resolutions eliminating boundary points as a metric of proof quality. We argue that obtaining proofs with a high value of this metric requires taking into account the formula structure. We show that for any unsatisfiable CNF formula there always exists a proof consisting only of resolutions eliminating cut boundary points (which are a relaxation of the notion of boundary points). This result enables building resolution SAT-solvers that are driven by elimination of cut boundary points.

This chapter is an extended version of the conference paper [9].

7.1 Introduction

Resolution-based SAT-solvers [3, 6, 10, 12, 13, 15, 16] have achieved great success in numerous applications. However, the reason for this success and, more generally, the semantics of resolution are not well understood yet. This obviously impedes
progress in SAT-solving. In this chapter, we study the relation between the resolution proof system [2] and boundary points [11]. The most important property of boundary points is that they mandate particular resolutions of a proof. So by studying the relation between resolution and boundary points one gets a deeper understanding of resolutions proofs, which should lead to building better SAT-solvers.

Given a CNF formula $F$, a non-satisfying complete assignment $p$ is called an $l(x)$-boundary point, if it falsifies only the clauses of $F$ that have the same literal $l(x)$ of variable $x$. The name is due to the fact that for satisfiable formulas the set of such points contains the boundary between satisfying and unsatisfying assignments. If $F$ is unsatisfiable, for every $l(x)$-boundary point $p$ there is a resolvent of two clauses of $F$ on variable $x$ that eliminates $p$. (That is, after adding such a resolvent to $F$, $p$ is not an $l(x)$-boundary point anymore.) On the contrary, for a non-empty satisfiable formula $F$, there is always a boundary point that cannot be eliminated by adding a clause implied by $F$.

To prove that a CNF formula $F$ is unsatisfiable it is sufficient to eliminate all its boundary points. In the resolution proof system, one reaches this goal by adding to $F$ resolvents. If formula $F$ has an $l(x)$-boundary point, a resolution proof has to have a resolution operation on variable $x$. The resolvents of a resolution proof eventually eliminate all boundary points. We will call a resolution mandatory if it eliminates a boundary point of the initial formula $F$ that has not been eliminated by adding the previous resolvents. (In [9] such a resolution was called boundary.)

Intuitively, one can use the Share of Mandatory Resolutions (SMR) of a proof as a metric of proof quality. The reason is that finding mandatory resolutions is not an easy task. (Identification of a boundary point is computationally hard, which implies that finding a mandatory resolution eliminating this point is not easy either.) However, finding mandatory resolutions becomes much simpler if one knows subsets of clauses of $F$ such that resolving clauses of these subsets eliminate boundary points. (An alternative is to try to guess these subsets heuristically.) Intuitively, such subsets have a lot to do with the structure of the formula. So the value of SMR may be used to gauge how well the resolution proof built by a SAT-solver follows the structure of the formula.

We substantiate the intuition above experimentally by comparing two kinds of proofs for equivalence checking formulas. (These formulas describe equivalence checking of two copies of a combinational circuit.) Namely, we consider short proofs of linear size particularly tailored for equivalence checking formulas and much longer proofs generated by a SAT-solver with conflict-driven learning. We show experimentally that the share of boundary resolution operations in high-quality specialized proofs is much greater than in proofs generated by the SAT-solver.

Generally speaking, it is not clear yet if for any irredundant unsatisfiable formula there is a proof consisting only of mandatory resolutions. However, as we show in this chapter, for any unsatisfiable formula $F$ there always exists a proof where each resolution eliminates a cut boundary point. The latter is computed with respect to the CNF formula $F_T$ consisting of clauses of $F$ and their resolvents that specify a cut $T$ of a resolution graph describing a proof. (Formula $F_T$ is unsatisfiable for any cut $T$.) The notion of a cut boundary point is a relaxation of that of a boundary