Chapter 5
Semidynamical Systems and Delay Equations

Abstract An autonomous system of delay differential equations is shown to generate a semidynamical system on the space $C$ of continuous functions on the delay interval. Omega limit sets are defined and shown to have the same properties as for ODEs, with minor exceptions, although they are subsets of $C$. The dynamics of the delayed logistic equation and the chemostat model are treated in detail. A special class of delay equations is shown to generate monotone dynamics; solutions converge to equilibrium. Liapunov functions and the LaSalle invariance principle are used to study the dynamics of a delayed logistic equation with both instantaneous and delayed density dependence.

5.1 The Dynamical Systems Viewpoint

A dynamical system consists of a set $X$ of “states” and a rule $\Lambda$ describing how states change with time. $X$ is called the state space. If at time $s$ you are at state $x$ and later, at time $t$, find yourself at state $x'$, then $x' = \Lambda(t,s,x)$. The transition rule (function) should take as inputs the “initial” time $s$ and state $x$ and produce the new state $x'$ at time $t$. The function $\Lambda$ should have some properties consistent with this interpretation of its meaning. For example, it should obviously satisfy

$$\Lambda(s,s,x) = x, \forall s, x \tag{5.1}$$

If at time $s$ we are at state $x$, at time $r$ we are at state $x'$, and at time $t$ are at state $x''$, then $x' = \Lambda(r,s,x)$, $x'' = \Lambda(t,r,x')$, and $x'' = \Lambda(t,s,x)$ should hold in keeping with our interpretation. It follows that $\Lambda$ should satisfy

$$\Lambda(t,s,x) = \Lambda(t,r,\Lambda(r,s,x)), \forall t, s, r, x \tag{5.2}$$

So far, our arguments have been informal and, in particular, we have not specified from what set one should take the “time” $t$. For discrete-time dynamical systems, we might choose this set to be the integers $\mathbb{Z}$, or the nonnegative integers $\mathbb{Z}_+$. If, for
example, we are given a sequence of maps $F_n : X \to X$ for $n \in \mathbb{Z}$ we may consider the dynamics generated by the recursion

$$x_{n+1} = F_n(x_n), n \geq s, x_s = x$$

where $s \in \mathbb{Z}$ is the initial time and $x$ the initial state. This generates the sequence:

$$x_s = x \to x_{s+1} = F_s(x_s) \to x_{s+2} = F_{s+1}(x_{s+1}) \to \cdots$$

The map $F_s$ is applied at time $s$ to get the state at time $s + 1$. The transition rule is

$$\Lambda(t, s, x) = F_{t-s} \circ F_{t-s-2} \circ \cdots \circ F_s(x)$$

where $\circ$ denotes function composition.

If we are given a single map $F : X \to X$ we may consider the dynamics generated by

$$x_{n+1} = F(x_n), n \geq s, x_s = x$$

which is formally the same as above where $F_n$ is the constant sequence $F_n = F$. Define $F^{(p)}(x) = (F \circ F \circ F \cdots \circ F)(x)$ to be the $p$-fold composition of $F$ with itself where, in general, $p \in \mathbb{Z}_+$. Then we define $\Lambda(t, s, x) = F^{(t-s)}(x)$ for $x \in X$ and $s \in \mathbb{Z}$ but $t \in \mathbb{Z}$ must in general satisfy $t > s$ because $F$ need not be invertible; if $F$ is invertible, then no restriction on $t$ is necessary. A noninvertible map $F$ generates the simplest dynamical system where one sees clearly why we can generally not go backward in time.

For continuous-time dynamical systems, one may choose the reals or the nonnegative reals. The quintessential example of a dynamical system is that generated by a system of ODEs. Here, we typically are interested in solutions of the initial-value problem

$$x' = f(t, x), x(s) = x_0$$

Under suitable conditions, there is a unique solution $x(t)$, which we often write as $x(t, s, x_0)$ to remind ourselves that the solution depends on all three arguments. In this case, $\Lambda(t, s, x_0) = x(t, s, x_0)$. For ODEs there is no asymmetry between the past and the future so it is natural to take the real line as our time set.

We have noted already that delay differential equations can generally be solved only forward in time, and that we should expect a continuous-time dynamical system in this case. Therefore, we specialize our formal definition of a dynamical system with these features in mind. Define

$$S = \{(t, s) \in \mathbb{R} \times \mathbb{R} : t \geq s\}$$

We say that $\Lambda : S \times X \to X$ is a semidynamical system if it satisfies (5.1) and (5.2); for the latter, $(t, r)$ and $(r, s)$ must belong to $S$. The “semi” in semidynamical reflects the restriction that we may only go forward in time, that is, $t \geq s$ in the definition of $\Lambda(t, s, x)$. In defining a dynamical system, we replace $S$ above by $S = \mathbb{R} \times \mathbb{R}$ and