Chapter 5
Matrices with Real or Complex Entries

A matrix $M \in M_{n \times m}(K)$ is an element of a vector space of finite dimension $n^2$. When $K = \mathbb{R}$ or $K = \mathbb{C}$, this space has a natural topology, that of $K^{nm}$. Therefore we may manipulate such notions as open and closed sets, and continuous and differentiable functions.

5.1 Special Matrices

5.1.1 Hermitian Adjoint

When considering matrices with complex entries, a useful operation is complex conjugation $z \mapsto \overline{z}$. One denotes by $\overline{M}$ the matrix obtained from $M$ by conjugating the entries. We then define the Hermitian adjoint matrix of $M$ by

$$M^* := (\overline{M})^T = \overline{M^T}.$$ 

One has $m^*_{ij} = \overline{m_{ji}}$ and $\det M^* = \overline{\det M}$. The map $M \mapsto M^*$ is an antiisomorphism, which means that it is antilinear (meaning that $(\lambda M)^* = \overline{\lambda} M^*$) and bijective. In addition, we have the product formula

$$(MN)^* = N^*M^*.$$

If $M$ is nonsingular, this implies $(M^*)^{-1} = (M^{-1})^*$; this matrix is sometimes denoted $M^{-*}$.

The interpretation of the Hermitian adjoint is that if we endow $\mathbb{C}^n$ with the canonical scalar product

$$\langle x, y \rangle = \overline{x_1}y_1 + \cdots + \overline{x_n}y_n,$$

and with the canonical basis, then $M^*$ is the matrix of the adjoint $(u_M)^*$; that is,

$$\langle Mx, y \rangle = \langle x, M^* y \rangle, \quad \forall x, y \in \mathbb{C}^n.$$
5.1.2 Normal Matrices

Definition 5.1 A matrix $M \in \mathbf{M}_n(C)$ is normal if $M$ and $M^*$ commute: $M^*M = MM^*$.

If $M$ has real entries, this amounts to having $MM^T = M^T M$.

Because a square matrix $M$ always commutes with $M, -M$, or $M^{-1}$ (assuming that the latter exists), we can define sub-classes of normal matrices. The following statement serves also as a definition of such classes.

Proposition 5.1 The following matrices $M \in \mathbf{M}_n(C)$ are normal.

- Hermitian matrices, meaning that $M^* = M$
- Skew-Hermitian matrices, meaning that $M^* = -M$
- Unitary matrices, meaning that $M^* = M^{-1}$

The Hermitian, skew-Hermitian, and unitary matrices are thus normal. One verifies easily that $H$ is Hermitian (respectively, skew-Hermitian) if and only if $x^*Hx$ is real (respectively, pure imaginary) for every $x \in \mathbb{C}^n$.

For real-valued matrices, we have instead

Definition 5.2 A square matrix $M \in \mathbf{M}_n(\mathbb{R})$ is

- Symmetric if $M^T = M$
- Skew-symmetric if $M^T = -M$
- Orthogonal if $M^T = M^{-1}$

We denote by $\mathbf{H}_n$ the set of Hermitian matrices in $\mathbf{M}_n(\mathbb{C})$. It is an $\mathbb{R}$-linear subspace of $\mathbf{M}_n(\mathbb{C})$, but not a $\mathbb{C}$-linear subspace, because $iM$ is skew-Hermitian when $M$ is Hermitian. If $M \in \mathbf{M}_{n \times m}(\mathbb{C})$, the matrices $M + M^*$, $i(M^* - M)$, $MM^*$, and $M^*M$ are Hermitian. One sometimes calls $\frac{1}{2}(M + M^*)$ the real part of $M$ and denotes it $\Re M$. Likewise, $\frac{1}{2i}(M - M^*)$ is the imaginary part of $M$ and is denoted $\Im M$. Both are Hermitian and we have

$$M = \Re M + i\Im M.$$  

This terminology anticipates Chapter 10.

A matrix $M$ is unitary if $uM$ is an isometry, that is $\langle Mx, My \rangle \equiv \langle x, y \rangle$. This is equivalent to saying that $\|Mx\| \equiv \|x\|$. The set of unitary matrices in $\mathbf{M}_n(\mathbb{C})$ forms a multiplicative group, denoted by $\mathbf{U}_n$. Unitary matrices satisfy $|\det M| = 1$, because $\det M^*M = |\det M|^2$ for every matrix $M$ and $M^*M = I_n$ when $M$ is unitary. The set of unitary matrices whose determinant equals 1, denoted by $\mathbf{SU}_n$ is obviously a normal subgroup of $\mathbf{U}_n$.

A matrix with real entries is orthogonal (respectively, symmetric, skew-symmetric) if and only if it is unitary, Hermitian, or skew-Hermitian.

5.1.3 Matrices and Sesquilinear Forms

Given a matrix $M \in \mathbf{M}_n(\mathbb{C})$, the map