Chapter 6
Graphs, Part II: More Advanced Notions

6.1 Γ -Cycles, Cocycles, Cotrees, Flows, and Tensions

In this section, we take a closer look at the structure of cycles in a finite graph $G$. It turns out that there is a dual notion to that of a cycle, the notion of a cocycle. Assuming any orientation of our graph, it is possible to associate a vector space $F$ with the set of cycles in $G$, another vector space $T$ with the set of cocycles in $G$, and these vector spaces are mutually orthogonal (for the usual inner product). Furthermore, these vector spaces do not depend on the orientation chosen, up to isomorphism. In fact, if $G$ has $m$ nodes, $n$ edges, and $p$ connected components, we prove that $\dim F = n - m + p$ and $\dim T = m - p$. These vector spaces are the flows and the tensions of the graph $G$, and these notions are important in combinatorial optimization and the study of networks. This chapter assumes some basic knowledge of linear algebra.

Recall that if $G$ is a directed graph, then a cycle $C$ is a closed $e$-simple chain, which means that $C$ is a sequence of the form $C = (u_0, e_1, u_1, e_2, u_2, \ldots, u_{n-1}, e_n, u_n)$, where $n \geq 1$; $u_i \in V$; $e_i \in E$ and

$$u_0 = u_n; \quad \{s(e_i), t(e_i)\} = \{u_{i-1}, u_i\}, \ 1 \leq i \leq n \text{ and } e_i \neq e_j \text{ for all } i \neq j.$$

The cycle $C$ induces the sets $C^+$ and $C^-$ where $C^+$ consists of the edges whose orientation agrees with the order of traversal induced by $C$ and where $C^-$ consists of the edges whose orientation is the inverse of the order of traversal induced by $C$. More precisely,

$$C^+ = \{e_i \in C \mid s(e_i) = u_{i-1}, t(e_i) = u_i\}$$

and

$$C^- = \{e_i \in C \mid s(e_i) = u_i, t(e_i) = u_{i-1}\}.$$

For the rest of this section, we assume that $G$ is a finite graph and that its edges are
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Fig. 6.1 Graph $G_8$ named, $e_1, \ldots, e_n$.

**Definition 6.1.** Given any finite directed graph $G$ with $n$ edges, with every cycle $C$ is associated a *representative vector* $\gamma(C) \in \mathbb{R}^n$, defined so that for every $i$, with $1 \leq i \leq n$,

$$
\gamma(C)_i = \begin{cases} 
+1 & \text{if } e_i \in C^+ \\
-1 & \text{if } e_i \in C^- \\
0 & \text{if } e_i \notin C.
\end{cases}
$$

For example, if $G = G_8$ is the graph of Figure 6.1, the cycle

$$
C = (v_3, e_7, v_4, e_6, v_5, e_5, v_2, e_1, v_1, e_2, v_3)
$$

corresponds to the vector

$$
\gamma(C) = (-1, 1, 0, 0, -1, -1, 1).
$$

Observe that distinct cycles may yield the same representative vector unless they are simple cycles. For example, the cycles

$$
C_1 = (v_2, e_5, v_5, e_6, v_4, e_4, v_2, e_1, v_1, e_2, v_3, e_3, v_2)
$$

and

$$
C_2 = (v_2, e_1, v_1, e_2, v_3, e_3, v_2, e_5, v_5, e_6, v_4, e_4, v_2)
$$

yield the same representative vector

$$
\gamma = (-1, 1, 1, 1, 1, 1, 0).
$$

In order to obtain a bijection between representative vectors and “cycles”, we introduce the notion of a “$\Gamma$-cycle” (some authors redefine the notion of cycle and call “cycle” what we call a $\Gamma$-cycle, but we find this practice confusing).

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1 We use boldface notation for the edges in $E$ in order to avoid confusion with the edges occurring in a cycle or in a chain; those are denoted in italic.