The purpose of this chapter is to teach you certain basic tricks to find indefinite integrals. It is of course easier to look up integral tables, but you should have a minimum of training in standard techniques.

XI, §1. SUBSTITUTION

We shall formulate the analogue of the chain rule for integration.

Suppose that we have a function $u(x)$ and another function $f$ such that $f(u(x))$ is defined. (All these functions are supposed to be defined over suitable intervals.) We wish to evaluate an integral having the form

$$\int f(u) \frac{du}{dx} \, dx,$$

where $u$ is a function of $x$. We shall first work out examples to learn the mechanics for finding the answer.

Example 1. Find $\int (x^2 + 1)^3(2x) \, dx$.

Put $u = x^2 + 1$. Then $du/dx = 2x$ and our integral is in the form

$$\int f(u) \frac{du}{dx} \, dx,$$
the function \( f \) being \( f(u) = u^3 \). We abbreviate \((du/dx) \, dx\) by \( du \), as if we could cancel \( dx \). Then we can write the integral as

\[
\int f(u) \, du = \int u^3 \, du = \frac{u^4}{4} = \frac{(x^2 + 1)^4}{4}.
\]

We can check this by differentiating the expression on the right, using the chain rule. We get

\[
\frac{d}{dx} \left( \frac{(x^2 + 1)^4}{4} \right) = \frac{4}{4} (x^2 + 1)^3 2x = (x^2 + 1)^3 2x,
\]

as desired.

**Example 2.** Find \( \int \sin(2x)(2) \, dx \).

Put \( u = 2x \). Then \( du/dx = 2 \). Hence our integral is in the form

\[
\int \sin u \, du = \int -\cos u \, du = -\cos(2x).
\]

Observe that

\[
\int \sin(2x) \, dx \neq -\cos(2x).
\]

If we differentiate \(-\cos(2x)\), we get \(\sin(2x) \cdot 2\).

The integral in Example 2 could also be written

\[
\int 2 \sin(2x) \, dx.
\]

It does not matter, of course, where we place the 2.

**Example 3.** Find \( \int \cos(3x) \, dx \).

Let \( u = 3x \). Then \( du/dx = 3 \). There is no extra 3 in our integral. However, we can take a constant in and out of an integral. Our integral is equal to

\[
\frac{1}{3} \int 3 \cos(3x) \, dx = \frac{1}{3} \int \cos u \, du = \frac{1}{3} \sin u = \frac{1}{3} \sin(3x).
\]