Historically, the notion of the radical was a direct outgrowth of the notion of semisimplicity. It may be somewhat surprising, however, to remark that the radical was studied first in the context of nonassociative rings (namely, finite-dimensional Lie algebras) rather than associative rings. In the work of E. Cartan, the radical of a finite-dimensional Lie algebra $A$ (say over $\mathbb{C}$) is defined to be the maximal solvable ideal of $A$: it is obtained as the sum of all the solvable ideals in $A$. The Lie algebra $A$ is semisimple iff its radical is zero, i.e., iff it has no nonzero solvable ideals. Cartan characterized the semisimplicity of a Lie algebra in terms of the nondegeneracy of its Killing form, and showed that any semisimple Lie algebra is a finite direct sum of simple Lie algebras. Moreover, he classified the finite-dimensional simple Lie algebras (over $\mathbb{C}$). Therefore, the structure theory of finite-dimensional semisimple Lie algebras is completely determined.

The theory of semisimple rings we developed in the last chapter may be viewed as the analogue of Cartan’s theory in the context of associative rings. It was developed by Molien and Wedderburn for finite-dimensional (associative) algebras, and generalized later by Artin to rings satisfying the descending chain condition. In the last chapter, we based the development of this theory on the use of semisimple (or completely reducible) modules; this treatment is somewhat different from the original treatment of Wedderburn. In developing the theory of finite-dimensional algebras over a field, Wedderburn defined for every such algebra $A$ an ideal, $\text{rad } A$, which is the largest nilpotent ideal of $A$, i.e., the sum of all the nilpotent ideals of $A$. In parallel with Cartan’s theory, the (finite-dimensional) algebra $A$ is semisimple iff its radical is zero. Such an algebra $A$ is (uniquely) the direct product of a finite number of finite-dimensional simple algebras $A_i$, and each $A_i$ is
2. Jacobson Radical Theory

(uniquely) a matrix algebra over a finite-dimensional division algebra. This beautiful theory of Wedderburn laid the modern foundation for the study of the structure of finite-dimensional algebras. Artin extended Wedderburn’s theory to rings with the minimum condition (appropriately called Artinian rings). For such rings $R$, the sum of all nilpotent ideals in $R$ is nilpotent, so $R$ has a largest nilpotent ideal $rad \, R$, called the Wedderburn radical of $R$. As we saw in the last Chapter, Wedderburn’s theory of simple and semisimple algebras can be extended successfully to rings satisfying the descending chain condition on one-sided ideals.

What about rings which do not satisfy Artin’s descending chain condition? For these rings $R$, the sum of all nilpotent ideals need no longer be nilpotent; thus, $R$ may not possess a largest nilpotent ideal, and so we no longer have the notion of a Wedderburn radical (see Ex. 4.25). The problem of finding the appropriate generalization of Wedderburn’s radical for arbitrary rings remained untackled for almost forty years. Finally, in a fundamental paper in 1945, N. Jacobson initiated the general notion of the radical of an arbitrary ring $R$: by definition, the (Jacobson) radical, $\text{rad } R$, of $R$ is the intersection of the maximal left (or maximal right) ideals of $R$. For rings satisfying a one-sided minimum condition, the Jacobson radical agrees with the classical Wedderburn radical, so, in general, the former provides a good substitute for the latter. Ever since its inception, Jacobson’s general theory of the radical has proved to be fundamental for the study of the structure of rings. In this chapter, we shall present the basic definition and properties of the Jacobson radical, and study the behavior of the radical under certain changes of rings. In the next chapter, we shall apply this material to the representation theory of algebras and groups, and explain the basic connections between ring theory and group representation theory, with some applications to group theory itself.

Needless to say, this chapter is a beginning, not an end. Having defined the Jacobson radical for arbitrary rings, we are led to a more general notion of semisimplicity: a ring $R$ is called Jacobson (or J-) semisimple if $\text{rad } R = 0$. These J-semisimple rings generalize the semisimple rings in Chapter 1, and therefore should play an important role in the study of rings possibly not satisfying the descending chain condition. We shall try to develop this theme in more detail in Chapter 4. Also, there are several other radicals which can be defined for arbitrary rings, and which provide alternative generalizations of the Wedderburn radical. These other radicals may not be as fundamental as the Jacobson radical, but in one way or another, they reflect more accurately the structure of the nil (and nilpotent) ideals of the ring, so one might say that these other radicals resemble the Wedderburn radical more than does the Jacobson radical. However, we can do only one thing at a time. So, in this chapter, we focus our attention on the Jacobson radical; other kinds of radicals (upper and lower nilradicals and the Levitzki radical) will be taken up in a future chapter.