After a short introduction to probability and to random fields, we shall define homogeneous random flow, its energy, vorticity and dissipation spectra, and its Fourier transform.

2.1. Introduction to Probability Theory

We have to formalize the notion of a variable (e.g., a flow) whose values depend on an experiment. We need a set of possible experiments (= "sample space"), a collection of "events" (i.e., experiments with some definite outcome), and some way to assign a probability to these events.

Thus, we pick a space \( \Omega \) to be our sample space; at this point it is not specified any further.

Let \( \mathcal{B} \) be a collection of subsets of \( \Omega \) (our events), such that whenever \( A, B \in \mathcal{B} \), their union \( A \cup B \), and their complements \( CA, CB \in \mathcal{B} \), and \( \emptyset \) = the empty set \( \in \mathcal{B} \). (Thus, in particular, \( \Omega \in \mathcal{B} \).) In addition, whenever \( A_n \in \mathcal{B}, n = 1, \ldots, \infty, \cup A_n \in \mathcal{B} \). Such a collection is called a \( \sigma \)-algebra. Let \( \mu \) be a non-negative set function defined on a \( \sigma \)-algebra \( \mathcal{B} \) (i.e., a rule which to each member of \( \mathcal{B} \) assigns some non-negative number); let \( \mu \) satisfy the following conditions:

(i) \( \mu(\emptyset) = 0 \) (This will become the statement that the probability of an impossible event is zero.)
(ii) \( \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n) \) whenever \( A_n \cap A_m = \emptyset, n \neq m, \) and \( A_n \in \mathcal{B} \ \forall n. \)

\( \mu \) is called a probability measure if \( \mu(\Omega) = 1 \) (the probability of something happening is 1); if \( \mu(\Omega) = 1 \), then the triple \( (\Omega, \mathcal{B}, \mu) \) is called a probability space. We shall denote a probability measure by \( P. \)

**Example:** Let \( \Omega \) be the real line, \( \mathcal{B} \) the \( \sigma \)-algebra generated by the half-open sets (i.e., sets of the form \( \{ x | a < x \leq b \} \), \( a, b \) constants), their finite intersections, complements and unions. The sets in this \( \mathcal{B} \) are called Borel sets. Given a probability measure \( P \) on \( (\Omega, \mathcal{B}) \), define the function

\[
F(x) = P(-\infty < \omega \leq x, \ \omega \in \Omega).
\]

\( F \) is called the distribution function of \( P. \) We have from the definition:

(i) \( F \) is non-decreasing

(ii) \( F \) is continuous from the right, i.e.,

\[
\lim_{0<\varepsilon \to 0} F(x + \varepsilon) = F(x);
\]

(iii) \( F(+\infty) = 1, \ F(-\infty) = 0. \)

Conversely, \( F(x) \), if it satisfies conditions (i)-(iii), defines an appropriate set function on the intervals of the form \( a < x \leq b \), and can be extended to their finite intersections, complements and unions.

If \( F \) is differentiable, \( F'(x) \) is called the probability density of \( P \),

\[
F'(x)dx = P(x < \omega \leq x + dx, \ \omega \in \Omega).
\]

Let \( (\Omega, \mathcal{B}, P) \) be a probability space. Let \( \eta(\omega) \) be a real-valued function defined for \( \omega \in \Omega. \) Let \( \eta \) satisfy the following condition: For every Borel set \( S \) on the real line, the set \( \{ \omega | \eta(\omega) \in S \} \in \mathcal{B} \) [i.e., one can assign a probability to the event that \( \eta(\omega) \) has a numerical value in a certain set]. \( \eta(\omega) \) is called a random variable, i.e., a variable whose value depends on an experiment, with probability assigned to the event that it should assume certain values. The integral of \( \eta(\omega) \), if it exists, is called the expected value or mean of \( \eta \) and is denoted by \( \langle \eta \rangle: \)

\[
\langle \eta \rangle = \int_{\Omega} \eta(\omega)dP.
\]

Note that the random variable \( \eta \) induces a probability measure \( P_{\eta} \) on the real line through the equation

\[
P_{\eta}(S) = P(\{ \omega | \eta(\omega) \in S \}) , \ \ S = \text{Borel set}.
\]