CHAPTER 11

Functions Defined by Integrals; Improper Integrals

11.1. The Derivative of a Function Defined by an Integral; the Leibniz Rule

The solutions of problems in differential equations, especially those which arise in physics and engineering, are frequently given in terms of integrals. Most often either the integrand of the integral representing the solution is unbounded or the domain of integration is an unbounded set. In this chapter we develop rules for deciding when it is possible to interchange the processes of differentiation and integration—commonly known as differentiation under the integral sign. When the integrand becomes infinite at one or more points or when the interval of integration is infinite, a study of the convergence of the integral is needed in order to determine whether or not the differentiation process is allowable. We establish the required theorems for bounded functions and domains in this section and treat the unbounded case in Sections 11.2 and 11.3.

Let \( f \) be a function with domain a rectangle \( R = \{(x, t): a \leq x \leq b, c \leq t \leq d\} \) in \( \mathbb{R}^2 \) and with range in \( \mathbb{R}^1 \). Let \( I \) be the interval \( \{x: a \leq x \leq b\} \) and form the function \( \phi: I \to \mathbb{R}^1 \) by the formula

\[
\phi(x) = \int_c^d f(x, t) \, dt. \tag{11.1}
\]

We now seek conditions under which we can obtain the derivative \( \phi' \) by differentiation of the integrand in (11.1). The basic formula is given in the following result.
Theorem 11.1 (Leibniz's rule). Suppose that $f$ and $f_{1}$ are continuous on the rectangle $R$ and that $\phi$ is defined by (11.1). Then

$$\phi'(x) = \int_{c}^{d} f_{1}(x, t) \, dt, \quad a < x < b.$$ \hspace{1cm} (11.2)

PROOF. Form the difference quotient

$$\frac{\phi(x + h) - \phi(x)}{h} = \frac{1}{h} \int_{c}^{d} \left[ f(x + h, t) - f(x, t) \right] \, dt.$$ \hspace{1cm} (11.3)

Observing that

$$f(x + h, t) - f(x, t) = \int_{x}^{x+h} f_{1}(z, t) \, dz,$$

we have

$$\frac{\phi(x + h) - \phi(x)}{h} = \frac{1}{h} \int_{c}^{d} \int_{x}^{x+h} f_{1}(z, t) \, dz \, dt.$$ \hspace{1cm} (11.4)

Since $f_{1}$ is continuous on the closed, bounded set $R$, it is uniformly continuous there. Hence, if $\varepsilon > 0$ is given, there is a $\delta > 0$ such that

$$|f_{1}(z, t) - f_{1}(x, t)| < \frac{\varepsilon}{d - c}$$

for all $t$ such that $c \leq t \leq d$ and all $z$ such that $|z - x| < \delta$. We now use the artifice

$$\int_{c}^{d} f_{1}(x, t) \, dt = \frac{1}{h} \int_{c}^{d} \int_{x}^{x+h} f_{1}(x, t) \, dz \, dt,$$ \hspace{1cm} (11.5)

which is valid because $z$ is absent in the integrand on the right. Subtracting (11.4) from (11.3), we find

$$\left| \frac{\phi(x + h) - \phi(x)}{h} - \int_{c}^{d} f_{1}(x, t) \, dt \right| = \left| \int_{c}^{d} \left\{ \frac{1}{h} \int_{x}^{x+h} [f_{1}(z, t) - f_{1}(x, t)] \, dz \right\} \, dt \right|.$$ \hspace{1cm} (11.5)

Now if $|h|$ is so small that $|z - x| < \delta$ in the integrand on the right side of (11.5), it follows that

$$\left| \frac{\phi(x + h) - \phi(x)}{h} - \int_{c}^{d} f_{1}(x, t) \, dt \right| \leq \int_{c}^{d} \left| \frac{1}{h} \int_{x}^{x+h} \frac{\varepsilon}{d - c} \, dz \right| \, dt = \frac{\varepsilon}{d - c} \cdot (d - c) = \varepsilon.$$

Since $\varepsilon$ is arbitrary, the left side of the above inequality tends to 0 as $h \to 0$. Formula (11.2) is the result.