CHAPTER 15

Functions on Metric Spaces; Approximation

15.1. Complete Metric Spaces

We developed many of the basic properties of metric spaces in Chapter 6. A complete metric space was defined in Chapter 13 and we saw the importance of such spaces in the proof of the fundamental fixed point theorem (Theorem 13.2). This theorem, which has many applications, was used to prove the existence of solutions of ordinary differential equations and the Implicit function theorem. We now discuss in more detail functions whose domain is a metric space, and we prove convergence and approximation theorems which are useful throughout analysis.

We recall that a sequence of points \( \{ p_n \} \) in a metric space \( S \) is a Cauchy sequence if and only if for every \( \varepsilon > 0 \) there is an integer \( N \) such that \( d(p_m, p_n) < \varepsilon \) whenever \( m, n > N \). A complete metric space \( S \) is one with the property that every Cauchy sequence in \( S \) converges to a point in \( S \). We remind the reader that Theorem 3.14 shows that \( \mathbb{R}^1 \) is a complete metric space and that Theorem 13.1 establishes the completeness of \( \mathbb{R}^N \) for every positive integer \( N \).

The notion of compactness, defined in Section 6.4, plays an essential part in the study of metric spaces. Repeating the definition, we say that a set \( A \) in a metric space \( S \) is compact if and only if each sequence of points \( \{ p_n \} \) in \( A \) contains a subsequence which converges to a point in \( A \). Taking into account the definitions of completeness and compactness, we see that Theorem 6.23 implies that every compact metric space \( S \) is complete. Also, since a closed subset of a compact metric space is compact (Theorem 6.21), then such a subset, considered as a metric space in its own right, is complete.

Let \( S \) be any metric space and \( f \) a bounded, continuous function on \( S \) into \( \mathbb{R}^1 \). The totality of all such functions \( f \) forms a metric space, denoted \( C(S) \),
when we define the distance \( d \) between two such functions \( f \) and \( g \) by the formula

\[
d(f, g) = \sup_{x \in S} |f(x) - g(x)|.
\]

(15.1)

It is a simple matter to verify that \( d \) has all the properties of a metric on \( C(S) \). We conclude directly from Theorem 13.3 on the uniform convergence of sequences of continuous functions that for any metric space \( S \), the space \( C(S) \) is complete.

**Examples.** Let \( I \) be the interval \( \{x: a \leq x \leq b\} \) be a closed interval in \( \mathbb{R}^1 \). Then \( I \), considered as a metric space (with the metric of \( \mathbb{R}^1 \)), is complete. On the other hand, an open interval \( J \) is not a complete metric space since there are Cauchy sequences in \( J \) converging to \( a \) and \( b \), two points which are not in the space. The collection of all rational numbers in \( \mathbb{R}^1 \) is an example of a metric space which is not complete, with respect to the metric of \( \mathbb{R}^1 \). Let \( 0 \) be the function in \( C(S) \) which is identically zero, and define \( B(0, r) \) as the set of all functions \( f \) in \( C(S) \) such that \( d(f, 0) < r \). That is, \( B(0, r) \) is the ball with center 0 and radius \( r \). Then \( B(0, r) \) is a metric space which is not complete. However, the set of all functions \( f \) which satisfy \( d(f, 0) \leq r \) with \( d \) given by (15.1) does form a complete metric space.

**Definitions.** Let \( S \) be a metric space. We say that a set \( A \) in \( S \) is **dense in** \( S \) if and only if \( A = S \). If \( A \) and \( B \) are sets in \( S \), we say that \( A \) is **dense in** \( B \) if and only if \( A \supset B \). We do not require that \( A \) be a subset of \( B \).

For example, let \( S \) be \( \mathbb{R}^1 \), let \( A \) be all the rational numbers, and let \( B \) be all the irrational numbers. Since \( A = \mathbb{R}^1 \) we see that \( A \supset B \) and so \( A \) is dense in \( B \). Note that \( A \) and \( B \) are disjoint.

**Definition.** Let \( S \) be a metric space and \( A \) a subset of \( S \). We say that \( A \) is **nowhere dense** in \( S \) if and only if \( \overline{A} \) contains no ball of \( S \). For example, if \( S \) is \( \mathbb{R}^1 \), then any finite set of points is nowhere dense. A convergent sequence of points and the set of integer points in \( \mathbb{R}^1 \) are other examples of nowhere dense sets. It is important to observe that being dense and nowhere dense are not complementary properties. It is possible for a set to be neither dense nor nowhere dense. For example, any bounded open or closed interval in \( \mathbb{R}^1 \) is neither dense nor nowhere dense in \( \mathbb{R}^1 \).

The next result, an equivalent formulation of the notion of a nowhere dense set, is useful in many applications.

**Theorem 15.1.** A set \( A \) is nowhere dense in a metric space \( S \) if and only if every open ball \( B \) of \( S \) contains an open ball \( B_1 \) which is disjoint with \( A \).