2 The Electromagnetic Field of a Known Charge Distribution

2.1 The Stationary-Action Principle and Conservation Theorems

If the field equations originate from a stationary-action principle, then a conserved current can be constructed for each parameter of an invariance group.

Field theory may be regarded as a generalization of the mechanics of point particles, in which the dynamical variables \( q_i(t) \) are replaced with fields \( \Phi(x, t) \), such as \( E(x, t) \) and \( B(x, t) \). The discrete index \( i \) goes over to the continuous variable \( x \), and, accordingly, the sum \( \sum_i \) is replaced with an integral \( \int d^3x \). A direct transcription of the formalism of I, §3, leads to infinite-dimensional manifolds, which we would prefer to avoid. Instead, we merely generalize the stationary-action principle (1:2.3.20) in order to find the analogues of the constants arising from the invariance properties. It is clear that in field theory the action \( \int dt \ L(q, \dot{q}) \) involves an integral over a four-dimensional submanifold \( N_4 \), and thus requires a 4-form, which allows the construction of a chart-independent integral.

The Lagrangian Formulation of Field Theory (2.1.1)

The action is given by

\[
W = \int_{N_4} \mathcal{L}(\Phi, d\Phi),
\]
where $\mathcal{L} \in E_4$ is the Lagrangian. The field equations result from the requirement that $\partial W = 0 \forall N_4$ compact and $\forall \Phi$ such that $\partial \Phi |_{\partial N_4} = 0.$

If we strengthen the homogeneous Maxwell equations to $F = dA$, then in pseudo-Riemannian space, the appropriate

**Electromagnetic Lagrangian** (2.1.2)

is

$$\mathcal{L} = -\frac{1}{2} dA \wedge *dA - A \wedge *J.$$  

**Proof**

Making a variation $A \rightarrow A + \delta A$ and using (1.2.18) (a), one finds

$$- \delta W = \int_{N_4} \delta A \wedge [\*J + d* dA] + \int_{\partial N_4} \delta A \wedge *dA,$$

which vanishes if $\delta A |_{\partial N_4} = 0$ and $d* F = -\*J$.

**Remarks** (2.1.3)

1. The variational formulation offers no guarantee of existence or uniqueness of the solutions of the field equations. Nowhere has it been assumed that $d* J = 0$, though without this condition it is not possible to satisfy $\delta W = 0 \forall \delta A$ such that $\delta A |_{\partial N_4} = 0$. The reason is easy to discover. With the gauge transformation $A \rightarrow A + d\Lambda$, where $\Lambda |_{\partial N_4} = 0$, $W$ changes by $\int_{N_4} \Lambda d*J$, and is linear in $\Lambda$ not only for infinitesimal $\Lambda$. As a linear functional, either $W$ has no stationary points, or else, if $d* J = 0$, it has a plateau. Accordingly, either there are no solutions at all, or else the solution is not uniquely fixed by any boundary condition whatsoever, because there is always the possibility of a gauge transformation.

2. According to (I: 5.2.8),

$$-\frac{1}{2} F \wedge *F = -\frac{1}{4} F_{\sigma \rho} F^{\sigma \rho} *1 = \frac{1}{2} (|E|^2 - |B|^2) *1.$$  

The sign of $\mathcal{L}$ has been chosen so that the interaction

$$-A \wedge *J = -iJ A = -J^2 A *1$$

of a point particle moving along the world-line $x(s)$ (cf. (1.3.25; 2)) has the same sign

$$-e \int_{-\infty}^{\infty} ds \dot{z}^\nu(s) A_\nu(z(s)) \delta^4 (x - z(s)) *1$$

† We use the symbol $\delta$ in § 2.1 for variations, rather than for codifferentials.