Chapter 2

Portfolio Management and the Capital Asset Pricing Model

In this chapter, we explore the issue of risk management in a portfolio of assets. The main issue is how to balance a portfolio, that is, how to choose the percentage (by value) of each asset in the portfolio so as to minimize the overall risk for a given expected return. The first lesson that we will learn is that the risks of each asset in a portfolio alone do not present enough information to understand the overall risk of the entire portfolio. It is necessary that we also consider how the assets interact, as measured by the covariance (or equivalently the correlation) of the individual risks.

2.1 Portfolios, Returns and Risk

For our model, we will assume that there are only two time periods: the initial time \( t = 0 \) and the final time \( t = T \). Each asset \( a_i \) has an initial value \( V_{i,0} \) and a final value \( V_{i,T} \).

Portfolios

A portfolio consists of a collection of assets \( a_1, \ldots, a_n \) in a given proportion. Formally, we define a portfolio to be an ordered \( n \)-tuple of real numbers

\[ \Theta = (\theta_1, \ldots, \theta_n) \]

where \( \theta_i \) is the number of units of asset \( a_i \). If \( \theta_i \) is negative then the portfolio has a short position on that asset: a short sale of stock, a short put or call and so on. A positive value of \( \theta_i \) indicates a long position: an owner of a stock, long on a put or call and so on.

Asset Weights

It is customary to measure the amount of an asset within a portfolio by its percentage by value. The weight \( w_i \) of asset \( a_i \) is the percentage of the value of the asset contained in the portfolio at time \( t = 0 \), that is,

\[ w_i = \frac{\theta_i V_{i,0}}{\sum_{j=1}^{n} \theta_j V_{j,0}} \]
Note that the sum of the weights will always be 1:
\[ w_1 + \cdots + w_n = 1 \]

**Asset Returns**

The return \( R_i \) on asset \( a_i \) is defined by the equation
\[ V_{i,T} = V_{i,0}(1 + R_i) \]
which is equivalent to
\[ R_i = \frac{V_{i,T} - V_{i,0}}{V_{i,0}} \]

Since the value of an asset at time \( T \) in the future is a random variable, so is the return \( R_i \). Thus, we may consider the expected value and the variance of the return. The expected return of asset \( a_i \) is denoted by
\[ \mu_i = \mathbb{E}(R_i) \]

The variance of the return of asset \( a_i \)
\[ \sigma^2_i = \text{Var}(R_i) \]
is called the risk of asset \( a_i \). We will also consider the standard deviation as a measure of risk when appropriate.

**Portfolio Return**

The return on the portfolio itself is defined to be the weighted sum of the returns of each asset
\[ R = \sum_{i=1}^{n} w_i R_i \]
For instance, suppose that a portfolio has only 2 assets, with weights 0.4 and 0.6 and returns equal to 10% and 8%, respectively. Then the return on the portfolio is
\[(0.4)(0.10) + (0.6)(0.08) = 0.088 = 8.8\% \]
Since the expected value operator is linear, the expected return of the portfolio as a whole is
\[ \mu = \sum_{i=1}^{n} w_i \mu_i \]