THE ELEMENTARY PROOF OF THE PRIME NUMBER THEOREM: AN HISTORICAL PERSPECTIVE

(by D. Goldfeld)

The study of the distribution of prime numbers has fascinated mathematicians since antiquity. It is only in modern times, however, that a precise asymptotic law for the number of primes in arbitrarily long intervals has been obtained. For a real number $x > 1$, let $\pi(x)$ denote the number of primes less than x. The prime number theorem is the assertion that

$$\lim_{x\to\infty} \pi(x) \Big/ \frac{x}{\log(x)} = 1.$$

This theorem was conjectured independently by Legendre and Gauss.

The approximation

$$\pi(x) = \frac{x}{A\log(x) + B}$$

was formulated by Legendre in 1798 **[Le1]** and made more precise in **[Le2]** where he provided the values $A = 1, B = -1.08366$. On August 4, 1823 (see **[La1]**, page 6) Abel, in a letter to Holmboe, characterizes the prime number theorem (referring to Legendre) as perhaps the most remarkable theorem in all mathematics.

Gauss, in his well known letter to the astronomer Encke, (see **[La1]**, page 37) written on Christmas eve 1849 remarks that his attention to the problem of finding an asymptotic formula for $\pi(x)$ dates back to 1792 or 1793 (when he was fifteen or sixteen), and at that time noticed that the density of primes in a chiliad (i.e. $[x, x+1000]$) decreased approximately as $1/\log(x)$ leading to the approximation

$$\pi(x) \approx \mathrm{Li}(x) = \int_2^x \frac{dt}{\log(t)}.$$

The remarkable part is the continuation of this letter, in which he said (referring to Legendre's $\frac{x}{\log(x) - A(x)}$ approximation and Legendre's value $A(x) = 1.08366$) that whether the quantity $A(x)$ tends to 1 or to a limit close to 1, he does not dare conjecture.

The first paper in which something was proved at all regarding the asymptotic distribution of primes was Tchebychef's first memoir (**[Tch1]**) which was read before the Imperial Academy of St. Petersburg in 1848. In that paper Tchebychef proved that if any approximation to $\pi(x)$ held to order $x/\log(x)^N$ (with some fixed large positive integer N) then that approximation had to be $\mathrm{Li}(x)$. It followed from this that Legendre's conjecture that $\lim_{x\to\infty} A(x) = 1.08366$ was false, and that if the limit existed it had to be 1.

The first person to show that $\pi(x)$ has the order of magnitude $\frac{x}{\log(x)}$ was Tchebychef in 1852 **[Tch2]**. His argument was entirely elementary and made use of properties of factorials. It is easy to see that the highest power of a prime p which divides $x!$ (we assume x is an integer) is simply

$$\left[\frac{x}{p}\right] + \left[\frac{x}{p^2}\right] + \left[\frac{x}{p^3}\right] + \cdots$$

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where $[t]$ denotes the greatest integer less than or equal to t. It immediately follows that

$$x! = \prod_{p \le x} p^{[x/p]+[x/p^2]+\cdots}$$

and

$$\log(x!) = \sum_{p \le x} \left(\left[\frac{x}{p}\right] + \left[\frac{x}{p^2}\right] + \left[\frac{x}{p^3}\right] + \cdots \right) \log(p).$$

Now $\log(x!)$ is asymptotic to $x\log(x)$ by Stirling's asymptotic formula, and, since squares, cubes, ... of primes are comparatively rare, and $[x/p]$ is almost the same as x/p, one may easily infer that

$$x \sum_{p \le x} \frac{\log(p)}{p} = x\log(x) \; + \; O(x)$$

from which one can deduce that $\pi(x)$ is of order $\frac{x}{\log(x)}$. This was essentially the method of Tchebychef, who actually proved that **[Tch2]**

$$B \; < \; \pi(x) \Big/ \frac{x}{\log(x)} \; < \; \frac{6B}{5}$$

for all sufficiently large numbers x, where

$$B = \frac{\log 2}{2} + \frac{\log 3}{3} + \frac{\log 5}{5} - \frac{\log 30}{30} \; \approx \; 0.92129$$

and

$$\frac{6B}{5} \; \approx \; 1.10555.$$

Unfortunately, however, he was unable to prove the prime number theorem itself this way, and the question remained as to whether an elementary proof of the prime number theorem could be found.

Over the years there were various improvements on Tchebychef's bound, and in 1892 Sylvester **[Syl1]**, **[Syl2]** was able to show that

$$0.956 \; < \; \pi(x) \Big/ \frac{x}{\log(x)} \; < \; 1.045$$

for all sufficiently large x. We quote from Harold Diamond's excellent survey article **[D]**:

> The approach of Sylvester was *ad hoc* and computationally complex; it offered no hope of leading to a proof of the P.N.T. Indeed, Sylvester concluded in his article with the lament that "...we shall probably have to wait [for a proof of the P.N.T.] until someone is born into the world so far surpassing Tchebychef in insight and penetration as Tchebychef has proved himself superior in these qualities to the ordinary run of mankind."