Chapter 23
Resolvents of Monotone Operators

Two quite useful single-valued, Lipschitz continuous operators can be associated with a monotone operator, namely its resolvent and its Yosida approximation. This chapter is devoted to the investigation of these operators. It exemplifies the tight interplay between firmly nonexpansive mappings and monotone operators. Indeed, firmly nonexpansive operators with full domain can be identified with maximally monotone operators via resolvents and the Minty parametrization. When specialized to subdifferential operators, resolvents become proximity operators. Numerous calculus rules for resolvents are derived. Finally, we address the problem of finding a zero of a maximally monotone operator, via the proximal-point algorithm and via approximating curves.

23.1 Definition and Basic Identities

Definition 23.1 Let \( A : H \to 2^H \) and let \( \gamma \in \mathbb{R}^{++} \). The resolvent of \( A \) is

\[
J_A = (\text{Id} + A)^{-1}
\]

(23.1)

and the Yosida approximation of \( A \) of index \( \gamma \) is

\[
\gamma A = \frac{1}{\gamma} \left( \text{Id} - J_\gamma A \right).
\]

(23.2)

The following properties follow at once from the above definition and (1.7).

Proposition 23.2 Let \( A : H \to 2^H \), let \( \gamma \in \mathbb{R}^{++} \), let \( x \in H \), and let \( p \in H \). Then the following hold:

(i) \( \text{dom} J_\gamma A = \text{dom} \gamma A = \text{ran}(\text{Id} + \gamma A) \) and \( \text{ran} J_\gamma A = \text{dom} A \).
(ii) \( p \in J_\gamma A x \iff x \in p + \gamma A p \iff x - p \in \gamma A p \iff (p, \gamma^{-1}(x - p)) \in \text{gra} A \).
(iii) \( p \in \gamma A x \iff p \in A(x - \gamma p) \iff (x - \gamma p, p) \in \text{gra} A \).
Example 23.3 Let \( f \in \Gamma_0(\mathcal{H}) \) and let \( \gamma \in \mathbb{R}_{++} \). Then Proposition 16.34 yields
\[
J_{\gamma \partial f} = \text{Prox}_{\gamma f}.
\]
(23.3)

In turn, it follows from Proposition 12.29 that the Yosida approximation of the subdifferential of \( f \) is the Fréchet derivative of the Moreau envelope; more precisely,
\[
\gamma(\partial f) = \nabla(\gamma f).
\]
(23.4)

Example 23.4 Let \( C \) be a nonempty closed convex subset of \( \mathcal{H} \) and let \( \gamma \in \mathbb{R}_{++} \). Setting \( f = \iota_C \) in Example 23.3 and invoking Example 12.25 and \( \iota \) is the Fréchet derivative of the Moreau envelope; more precisely,
\[
J_{N_C} = (\text{Id} + N_C)^{-1} = \text{Prox}_{\iota C} = P_C \quad \text{and} \quad \gamma_{N_C} = \frac{1}{\gamma}(\text{Id} - P_C).
\]
(23.5)

Example 23.5 Let \( \mathcal{H} \) be a real Hilbert space, let \( x_0 \in \mathcal{H} \), suppose that \( \mathcal{H} = L^2([0, T]; \mathbb{H}) \), and let \( A \) be the time-derivative operator (see Example 2.9)
\[
A: \mathcal{H} \to 2^{\mathcal{H}}: x \mapsto \begin{cases} \{x'\}, & \text{if } x \in W^{1,2}([0, T]; \mathbb{H}) \text{ and } x(0) = x_0; \\ \emptyset, & \text{otherwise.} \end{cases}
\]
(23.6)

Then \( \text{dom} \ J_A = \mathcal{H} \) and, for every \( x \in \mathcal{H} \),
\[
J_Ax: [0, T] \to \mathbb{H}: \ t \mapsto e^{-t}x_0 + \int_0^t e^{s-t}x(s)ds.
\]
(23.7)

Proof. Let \( x \in \mathcal{H} \) and set \( y: t \mapsto e^{-t}x_0 + \int_0^t e^{s-t}x(s)ds \). As shown in Proposition 21.3 and its proof, \( A \) is maximally monotone; hence \( \text{dom} \ J_A = \mathcal{H} \) by Theorem 21.1, and \( y \in W^{1,2}([0, T]; \mathbb{H}), \ y(0) = x_0, \) and \( x(t) = y(t) + y'(t) \) a.e. on \( ]0, T[ \). Thus, \( x = (\text{Id} + A)y \) and we deduce that \( y \in J_Ax \). Now let \( z \in J_Ax \), i.e., \( x = z + Az \). Then, by monotonicity of \( A, 0 = \langle y - z \mid x - x \rangle = \|y - z\|^2 + \langle y - z \mid Ay - Az \rangle \geq \|y - z\|^2 \) and therefore \( z = y \). \( \square \)

Proposition 23.6 Let \( A: \mathcal{H} \to 2^\mathcal{H}, \) let \( \gamma \in \mathbb{R}_{++}, \) and let \( \mu \in \mathbb{R}_{++} \). Then the following hold:

(i) \( \text{gra} \ \gamma A \subset \text{gra} (A \circ J_{\gamma A}) \).
(ii) \( \gamma A = (\text{Id} + A^{-1})^{-1} = (J_{\gamma A}A^{-1}) \circ \gamma^{-1} \text{Id} \).
(iii) \( \gamma \circ \mu A = \gamma (\mu A) \).
(iv) \( J_{\gamma (\mu A)} = \text{Id} + \gamma/(\gamma + \mu) (J_{\gamma + \mu}A - \text{Id}) \).

Proof. Let \( x \) and \( u \) be in \( \mathcal{H} \).

(i): We derive from (23.2) and Proposition 23.2(ii) that \( (x, u) \in \text{gra} \ \gamma A \Rightarrow \exists p \in J_{\gamma A}x \) \( u = \gamma^{-1}(x - p) \in Ap \Rightarrow u \in A(J_{\gamma A}x) \).

(ii): \( u \in \gamma Ax \Leftrightarrow \gamma u \in x - J_{\gamma A}x \Leftrightarrow x - \gamma u \in J_{\gamma A}x \Leftrightarrow x \in x - \gamma u + \gamma A(x - \gamma u) \Leftrightarrow u \in A(x - \gamma u) \Leftrightarrow x \in (\gamma u + \gamma^{-1}u) \Leftrightarrow u \in (\gamma \text{Id} + A^{-1})^{-1}x. \) Moreover, \( x \in \gamma u + A^{-1}u \Leftrightarrow \gamma^{-1}x \in u + \gamma^{-1}A^{-1}u \Leftrightarrow u \in J_{\gamma A}(\gamma^{-1}x) \).