Large $N$ Asymptotics in Random Matrices
The Riemann–Hilbert Approach

Alexander R. Its
Department of Mathematics Sciences, Indiana University–Purdue University
Indianapolis, Indianapolis, IN 46202-3216, USA itsa@math.iupui.edu

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5.1 The RH Representation of the Orthogonal Polynomials
and Matrix Models

5.1.1 Introduction

5.1.1.1 Hermitian Matrix Model

The Hermitian matrix model is defined as the ensemble $\mathcal{H}_N$ of random
Hermitian $N \times N$ matrices $M = (M_{ij})_{i,j=1}^N$ with the probability distribution

$$\mu_N (dM) = \hat{Z}_N^{-1} \exp(-N \operatorname{Tr} V(M)) \, dM .$$

(5.1)

Here the (Haar) measure $dM$ is the Lebesgue measure on $\mathcal{H}_N \equiv \mathbb{R}^{N^2}$, i.e.,

$$dM = \prod_j dM_{jj} \prod_{j<k} dM_{jk}^R dM_{jk}^I, \quad M_{jk} = M_{jk}^R + iM_{jk}^I .$$

The exponent $V(M)$ is a polynomial of even degree with a positive leading
coefficient,
\[
V(z) = \sum_{j=1}^{2m} t_j z^j, \quad t_{2m} > 0,
\]
and the normalization constant \( \hat{Z}_N \), which is also called \textit{partition function}, is given by the equation,
\[
\hat{Z}_N = \int_{\mathcal{H}_N} \exp(-N \text{Tr} V(M)) \, dM,
\]
so that,
\[
\int_{\mathcal{H}_N} \mu_N(dM) = 1.
\]
The model is also called a \textit{unitary ensemble}. The use of the word “unitary” refers to the invariance properties of the ensemble under unitary conjugation. The special case when \( V(M) = M^2 \) is called the \textit{Gaussian Unitary Ensemble} (GUE). (we refer to the book \cite{40} as a basic reference for random matrices; see also the more recent survey \cite{45} and monograph \cite{12}).

\subsection*{5.1.1.2 Eigenvalue Statistics}

Let \( z_0(M) < \cdots < z_N(M) \) be the ordered eigenvalues of \( M \). It is a basic fact (see e.g. \cite{45} or \cite{12}) that the measure (5.1) induces a probability measure on the eigenvalues of \( M \), which is given by the expression
\[
\frac{1}{Z_N} \prod_{1 \leq j < k \leq N} (z_j - z_k)^2 \exp\left(-N \sum_{j=1}^N V(z_j)\right) \, dz_0 \ldots dz_N,
\]
where the reduced partition function \( Z_N \) is represented by the multiple integral
\[
Z_N = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1 \leq j < k \leq N} (z_j - z_k)^2 \exp\left(-N \sum_{j=1}^N V(z_j)\right) \, dz_0 \ldots dz_N.
\]
The principal object of interest in the random matrix theory is the \textit{m-point correlation function} \( K_{Nm}(z_0 \cdots z_m) \) which is defined by the relation
\[
K_{Nm}(z_0 \cdots z_m) \, dz_0 \ldots dz_m
= \text{the joint probability to find the } k\text{th eigenvalue in the interval } [z_k, z_k + dz_k], \, k = 1, \ldots, m.
\]
The principal issue is the \textit{universality properties} of the random matrix ensembles. More specifically, this means the analysis of the \textit{m-point correlation}