Chapter 3
Generalized Additive Cauchy Equations

It is very natural for one to try to transform the additive Cauchy equation into other forms. Some typically generalized additive Cauchy equations will be introduced. The functional equation \( f(ax + by) = af(x) + bf(y) \) appears in Section 3.1. The Hyers–Ulam stability problem is discussed in connection with a question of Th. M. Rassias and J. Tabor. In Section 3.2, the functional equation (3.3) is introduced, and the Hyers–Ulam–Rassias stability for this equation is also studied. The stability result for this equation will be used to answer the question of Rassias and Tabor cited above. The last section deals with the functional equation \( f(x + y)^2 = (f(x) + f(y))^2 \). The continuous solutions and the Hyers–Ulam stability for this functional equation will be investigated.

3.1 Functional Equation \( f(ax + by) = af(x) + bf(y) \)

It is a very natural thing to generalize the additive Cauchy equation (2.1) into the functional equation \( f(ax + by) = af(x) + bf(y) \) and study the stability problems for this equation.

In the paper [312], Th. M. Rassias and J. Tabor asked whether the functional equation \( f(ax + by + c) = Af(x) + Bf(y) + C \) with \( abAB \neq 0 \) is stable in the sense of Hyers, Ulam, and Rassias.

C. Badea [8] answered this question of Rassias and Tabor for the case when \( c = C = 0, a = A, \) and \( b = B \).

**Theorem 3.1 (Badea).** Let \( a \) and \( b \) be nonnegative real numbers with \( \alpha = a + b > 0 \). Let \( H : [0, \infty)^2 \to [0, \infty) \) be a function for which there exists a positive number \( k < \alpha \) such that \( H(\alpha s, \alpha t) \leq kH(s, t) \) for all \( s, t \in [0, \infty) \). Given a real normed space \( E_1 \) and a real Banach space \( E_2 \), assume that a function \( f : E_1 \to E_2 \) satisfies the inequality

\[
\| f(ax + by) - af(x) - bf(y)\| \leq H(\|x\|, \|y\|) \\
(3.1)
\]

for all $x, y \in E_1$. Then there exists a unique function $A : E_1 \to E_2$ such that

$$A(ax + by) = aA(x) + bA(y)$$

for any $x$ and $y$ in $E_1$, and

$$\| f(x) - A(x) \| \leq (\alpha - k)^{-1} H(\|x\|, \|x\|)$$

(3.2)

for all $x \in E_1$.

Proof. Let $x$ be a fixed point of $E_1$. Putting $y = x$ in the inequality (3.1) yields

$$\| f(\alpha x) - \alpha f(x) \| \leq H(\|x\|, \|x\|)$$

which implies

$$\| (1/\alpha) f(\alpha x) - f(x) \| \leq (1/\alpha) H(\|x\|, \|x\|). \quad (a)$$

We now use an induction argument to prove

$$\| \alpha^{-n} f(\alpha^n x) - f(x) \| \leq \frac{1 - (k/\alpha)^n}{\alpha - k} H(\|x\|, \|x\|)$$

(b)

for any $n \in \mathbb{N}$. Assume that the inequality (b) holds true for a fixed $n \geq 1$. If we substitute $\alpha x$ for $x$ in (b) and divide the resulting inequality by $\alpha$, then we obtain

$$\| \alpha^{-n-1} f(\alpha^{n+1} x) - (1/\alpha) f(\alpha x) \| \leq \frac{1 - (k/\alpha)^n}{\alpha(\alpha - k)} H(\|x\|, \|x\|)$$

$$\leq \frac{1 - (k/\alpha)^n}{\alpha(\alpha - k)} k H(\|x\|, \|x\|).$$

This, together with (a), leads to the inequality

$$\| \alpha^{-n-1} f(\alpha^{n+1} x) - f(x) \| \leq \left( \frac{1 - (k/\alpha)^n}{\alpha(\alpha - k)} k + \frac{1}{\alpha} \right) H(\|x\|, \|x\|).$$

It can be easily seen that

$$\frac{1 - (k/\alpha)^{n+1}}{\alpha - k} = \frac{1 - (k/\alpha)^n}{\alpha(\alpha - k)} k + \frac{1}{\alpha}$$

which implies the validity of the inequality (b) for every $n \in \mathbb{N}$.

Let $m$ and $n$ be integers with $m > n > 0$. Then we have

$$\| \alpha^{-m} f(\alpha^m x) - \alpha^{-n} f(\alpha^n x) \| = \alpha^{-n} \| \alpha^{-(m-n)} f(\alpha^m x) - f(\alpha^n x) \|$$

$$= \alpha^{-n} \| \alpha^{-r} f(\alpha^r y) - f(y) \|,$$