Chapter 8
Quadratic Functional Equations

So far, we have discussed the stability problems of functional equations in connection with additive or linear functions. In this chapter, the Hyers–Ulam–Rassias stability of quadratic functional equations will be proved. Most mathematicians may be interested in the study of the quadratic functional equation since the quadratic functions are applied to almost every field of mathematics. In Section 8.1, the Hyers–Ulam–Rassias stability of the quadratic equation is surveyed. The stability problems for that equation on a restricted domain are discussed in Section 8.2, and the Hyers–Ulam–Rassias stability of the quadratic functional equation will be proved by using the fixed point method in Section 8.3. In Section 8.4, the Hyers–Ulam stability of an interesting quadratic functional equation different from the “original” quadratic functional equation is proved. Finally, the stability problem of the quadratic equation of Pexider type is discussed in Section 8.5.

8.1 Hyers–Ulam–Rassias Stability

The quadratic function \( f(x) = cx^2 \) (\( x \in \mathbb{R} \)), where \( c \) is a real constant, clearly satisfies the equation

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y).
\]

Hence, the equation (8.1) is called the *quadratic functional equation*.

There are a number of functional equations considered as quadratic and one of them will be introduced in Section 8.4. A *quadratic function* implies a solution of the quadratic functional equation (8.1).

A function \( f : E_1 \to E_2 \) between real vector spaces is a quadratic function if and only if there exists a symmetric biadditive function \( B : E_1^2 \to E_2 \) such that \( f(x) = B(x, x) \). (A function \( B : E_1^2 \to E_2 \) is called *biadditive* if and only if \( B \) is additive in each variable.) If \( f \) is a quadratic function, then the biadditive function \( B \) is sometimes called the *polar* of \( f \) and given by

\[
B(x, y) = (1/4)(f(x + y) - f(x - y)).
\]
F. Skof [331] was the first person to prove the Hyers–Ulam stability of the quadratic functional equation (8.1) for functions $f : E_1 \to E_2$ where $E_1$ and $E_2$ are a normed space and a Banach space, respectively. P. W. Cholewa [70] demonstrated that Skof’s theorem is also valid if $E_1$ is replaced with an abelian group $G$.

**Theorem 8.1 (Skof).** Let $G$ be an abelian group and let $E$ be a Banach space. If a function $f : G \to E$ satisfies the inequality

$$\| f(x + y) + f(x - y) - 2f(x) - 2f(y) \| \leq \delta$$

for some $\delta \geq 0$ and for all $x, y \in G$, then there exists a unique quadratic function $Q : G \to E$ such that

$$\| f(x) - Q(x) \| \leq (1/2)\delta$$

for any $x \in G$.

I. Fenyő [102] improved Theorem 8.1 by replacing the bound $(1/2)\delta$ with the best possible one, $(1/3)(\delta + \| f(0) \|)$. The proof of Theorem 8.1 is a special case of that of Theorem 8.3 by S. Czerwik [87], and hence, it will be omitted. Before starting the theorem of Czerwik, we need a lemma provided by the same author.

**Lemma 8.2.** Let $E_1$ and $E_2$ be normed spaces. Assume that there exist $\delta, \theta \geq 0$ and $p \in \mathbb{R}$ such that a function $f : E_1 \to E_2$ satisfies the inequality

$$\| f(x + y) + f(x - y) - 2f(x) - 2f(y) \| \leq \delta + \theta(\|x\|^p + \|y\|^p)$$

(8.2)

for all $x, y \in E_1 \setminus \{0\}$. Then for $x \in E_1 \setminus \{0\}$ and $n \in \mathbb{N}$

$$\| f(2^n x) - 4^n f(x) \| \leq (1/3)(4^n - 1)(\delta + c) + 2 \cdot 4^n \theta \|x\|^p (1 + a + \cdots + a^{n-1})$$

(8.3)

and

$$\| f(x) - 4^n f(2^{-n} x) \| \leq (1/3)(4^n - 1)(\delta + c) + 2^{1-p} \theta \|x\|^p (1 + b + \cdots + b^{n-1}),$$

(8.4)

where $a = 2^{p-2}$, $b = 2^{2-p}$, and $c = \|f(0)\|$.

**Proof.** Putting $x = y \neq 0$ in (8.2) yields

$$\| f(2x) - 4f(x) \| \leq \|f(0)\| + \delta + 2\theta \|x\|^p$$

which proves (8.3) for $n = 1$. Assume now that (8.3) holds true for each $k \leq n$ and $x \in E_1 \setminus \{0\}$. Then, for $n + 1$, we have...