Chapter 12
Second-Order Symmetric Duality with Generalized Invexity

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Abstract A pair of second-order symmetric dual programs such as Wolfe-type and Mond–Wier-type are considered and appropriate duality results are established. Second-order $\rho - (\eta, \theta)$-bonvexity and $\rho - (\eta, \theta)$-boncavity of the kernel function are studied. It is also observed that for a particular kernel function, both these pairs of programs reduce to general nonlinear problem introduced by Mangasarian. Many examples and counterexamples are illustrated to justify our work.

12.1 Introduction

The study of second-order duality is useful due to the computational advantage over first-order duality as it gives bounds for the value of the objective function when approximations are used (see [7], [8], [10]). Symmetric duality in nonlinear programming in which the dual of dual is primal was introduced by Dorn [5]. Subsequently Dantzig Eisenberg, and Cottle [4] and Mond [11] significantly developed the notion of symmetric duality. Motivated by the concept of second and higher duality in nonlinear programming problems introduced by Mangasarian [8], several researchers [1, 9, 11, 12] have been working in this field. Mond [10] established Mangasarian’s duality relations assuming rather simple inequalities for the objective and constant function. Bector and Chandra [3] called the functions satisfying these inequalities bonvex/boncave. Mond [10] has further studied second-order symmetric dual programs.

In this chapter we study second-order symmetric duality for Wolfe and Mond–Weir-type problems under $\rho - (\eta, \theta)$-bonvexity and $\rho - (\eta, \theta)$-boncavity.
assumptions, respectively. Different duality results (weak, strong, converse) are established. Many examples and counterexamples are discussed to support the work.

12.2 Notation and Preliminaries

Let \( \mathbb{R}^n \) denote the \( n \)-dimensional Euclidean space. Let \( f(x,y) \) be a twice-differentiable function in \( \mathbb{R}^n \times \mathbb{R}^m \) and \( f_\alpha(\vec{x},\vec{y}) \) denote the gradient of \( f \) with respect to \( x \) at \((\vec{x},\vec{y})\); \( f_\gamma(\vec{x},\vec{y}) \) is defined similarly. Also let \( f_{xx}(\vec{x},\vec{y}) \) and \( f_{yx}(\vec{x},\vec{y}) \) denote the \( n \times n \) and \( m \times m \) symmetric Hessian matrices at \((\vec{x},\vec{y})\), respectively. The symbol \( z^T \) stands for the transpose of a vector \( z \).

**Definition 12.1** ([6]). A twice-differentiable function \( f \) defined on a set \( S \subseteq \mathbb{R}^n \) is said to be \( \eta \)-convex at \( \vec{x} \in S \) if there exists \( \eta(x,\vec{x}) \) defined on \( S \times S \) such that for all \( p \in \mathbb{R}^n \),

\[
f(x) - f(\vec{x}) \geq [\eta(x,\vec{x})]^T [\nabla f(\vec{x}) + \nabla^2 f(\vec{x}) p] - \frac{1}{2} p^T \nabla^2 f(\vec{x}) p, \quad \forall x \in S.
\]

**Definition 12.2** ([6]). A twice-differentiable function \( f \) defined on a set \( S \subseteq \mathbb{R}^n \) is said to be \( \eta \)-pseudo-convex at \( \vec{x} \in S \) if there exists \( \eta(x,\vec{x}) \) defined on \( S \times S \) such that for all \( p \in \mathbb{R}^n \),

\[
[\eta(x,\vec{x})]^T [\nabla f(\vec{x}) + \nabla^2 f(\vec{x}) p] \geq 0 \Rightarrow f(x) \geq f(\vec{x}) - \frac{1}{2} p^T \nabla^2 f(\vec{x}) p, \forall x \in S.
\]

A twice differentiable function is \( \eta \)-concave and \( \eta \)-pseudo-concave if \(-f\) is \( \eta \)-convex and \( \eta \)-pseudoconvex, respectively.

**Definition 12.3.** Let \( f(x,y) \) be a twice-differentiable function on \( \mathbb{R}^n \times \mathbb{R}^m \), \( f \) is said to be second-order \( \rho - (\eta,\theta) \)-convex at \( u \in \mathbb{R}^n \), for fixed \( v \), with respect to \( \eta, \theta \) if there exist \( \eta, \theta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), and \( \rho \in \mathbb{R} \) such that

\[
f(x,v) - f(u,v) \geq [\eta(x,u)]^T [f_u(u,v) + f_{uu}(u,v)r] - \frac{1}{2} r^T f_{uu}(u,v)r + \rho \|\theta(x,u)\|^2,
\]

\( \forall x, \ r \in \mathbb{R}^n \).

It follows that every \( \eta \)-convex function is \( \rho - (\eta,\theta) \)-bonvex but the converse is not true, which follows from the following counterexample (12.1).

**Example 12.1.** Let \( f : [0,2\pi] \times [0,2\pi] \rightarrow \mathbb{R} \) be defined by

\[
f(x,y) = -2x^2 - 2x - 2y - \sin^2 x.
\]

The above function is not \( \eta \)-convex but \( \rho - (\eta,\theta) \)-bonvex for \( \eta(x,u) = -\frac{1}{2} \sin^2 u - u - x - 1 \), \( \theta(x,u) = \sqrt{x^2 + ux + x + u + 12} \), and \( \rho = -100 \).