This topic is very substantial, but we will try to give the spirit of this important area in one short chapter.

4.1 Row Vectors

The notion of a vector is geometrically inspired. A geometric vector in our 3-dimensional world (or the 2-dimensional world of this page) is an arrow i.e. a straight line segment of finite length, with orientation. A geometric vector has direction but not position. Thus parallel geometric vectors of the same length, pointing in the same direction, are deemed to be equal.

The method of adding geometric vectors is “nose to tail” as in Figure 4.1. We need to liberate ourselves from the tyranny of pictures for at least two reasons. First, because they can be difficult to draw, and second because we must escape from the 2- and 3-dimensional prisons in which our geometric imaginations are trapped. We do this by capturing the geometric notion of a vector in purely algebraic terms.

To this end we define a row vector to be an element of $\mathbb{R}^n$. Thus a row vector is a finite sequence $(x_1, x_2, \ldots, x_n)$ of real numbers. Sequences of the same length can by added (or subtracted) co-ordinatewise. Thus

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \quad (4.1)$$
There is also a way of multiplying a row vector by a real number $\lambda$ according to the recipe
\[\lambda \cdot (x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n).\] (4.2)

We now show that this algebraic version of the theory of vectors captures the geometry correctly. We work in 3-dimensional space but 2- or 1-dimensional space would do just as well. Set up a co-ordinate system with an origin and mutually perpendicular axes. Calibrate the axes – which is a way of saying that you will regard each axis as a copy of the real line with 0 at the origin. Take any geometric vector $\mathbf{v}$ (we will write all vectors in bold type) and translate it until its tail is parked at the origin. Here translate means that you must not change the length, direction or orientation of the geometric vector when you move its tail to the origin. The co-ordinates of the tip of the vector are now at $(x_1, x_2, x_3)$. This sets up a bijection between geometric vectors and ordered triples of real numbers (i.e. $\mathbb{R}^3$). It is easy to check that the addition of row vectors exactly captures addition of geometric vectors. Thus you can add geometric vectors the geometric way, and then read off the row vector equivalent – or alternatively take the two geometric vectors to be added, turn them into elements of $\mathbb{R}^3$ by the specified procedure, and then add the row vectors using Equation (4.1). It doesn’t matter which you do, you get the same answer.

In this context, the real numbers are called scalars because of their rôle in Equation (4.2). The reason is that multiplication by a scalar quantity scales the length of the vector, without changing its direction. The orientation will reverse if you multiply by a negative real number. If you multiply any row vector by 0 you will obtain the zero vector $\mathbf{0} = (0,0,\ldots,0)$ which acts as an