Chapter 4

Liapunov—Schmidt Reduction

The theory and application of bifurcation are presented in this chapter and chapter 5. This chapter describes the basic concepts of bifurcation in ordinary differential equations, Liapunov—Schmidt reduction (LS reduction for short), singularity theory and applications of all these theories. Chapter 5 introduces the centre manifold theorem and the normal form of vector fields. Chapter 6 presents the Hopf bifurcation and double zero eigenvalues. Chapter 7 explains the applications of the averaging method in bifurcation theory.

4.1 Basic Concepts of Bifurcation

Bifurcation theory is concerned with the instability of the solution of ordinary differential equations arising from changes in parameters and the consequential variations in a number of solutions. We have discussed the structural stability of dynamical systems in section 3.5 of chapter 3. Bifurcations of dynamical systems (primarily bifurcation in ordinary differential equations) will be introduced in this chapter and discussed further in chapter 5 to 7. If a dynamical system is structurally unstable, a slight proper perturbation would cause a sudden change in quality of its topological structure. Such a change is bifurcation. Bifurcation always occurs in unstable structures and this confirms the close association of bifurcation with structural stability. The study of dynamical systems will be focused on the change of qualitative behaviour caused by structural instability as well as stability. Moreover, a series of bifurcations may give rise to chaotic motion in the dynamical system, and this indicates that bifurcation and chaos are closely related complicated motions. Being a comprehensive subject, the bifurcation of dynamical systems is an integral part of the study of dynamical systems and non-linear differential equations, and has broad prospects of application. Since the 18th century studies on the loss of stability in celestial, elastic and fluid mechanics, and non-linear oscillation, have come within the scope of bifurcation. By the 1970s, with the impetus from dynamical systems, non-linear analysis, non-linear differential equations, and the aid of highly effective numerical calculation methods, the mathematical theory of bifurcation has begun to take shape and find wide uses in mechanics, physics, chemistry, biology, ecology, cybernetics, numerical calculation, engineering, technology, economics and sociology. At present, the study of bifurcation is advancing in-depth and at high speed both in theory and application. In practice, since many systems contain one or more parameter, we determine whether or not the topological structure of a system changes when continuous slight changes in parameters take place. Let us examine a few examples first.
Example 1  Bifurcation of the buckling of a wire arch

Imagine an arch made of a metallic wire, Fig. 4.1(a) and (b). The wire arch can be used to demonstrate buckling caused by gravity. The equation of the arch in motion is

\[ \frac{d\theta}{dt} = F(l, \theta) \quad (4.1) \]

When the length \( l \) of the wire arch is very small, eq. (4.1) has only one stable solution \( \theta = 0 \) (expressing a perpendicular state), Fig. 4.1(b). When \( l < l_c \) (the critical value of the loss of stability), the position of leftward or rightward buckling of the arch is stable. When \( l_0 < l < l_c \), there may be three stable steady solutions: the upright one (\( \theta = 0 \)), the leftward buckling or rightward buckling (\( \theta \neq 0 \)). When \( l > l_c \), the arch cannot keep upright but inclines leftward or rightward. The relationship between the change in length of the wire and the number of positions can be expressed by a curve. This is the bifurcation diagram of the wire arch, Fig. 4.1(c).

The \( \theta \) in the figure is the state variable, and the length \( l \) of the wire is the bifurcating parameter. It can be seen from the bifurcation diagram that with variation of the magnitude of the wire, i.e., from large to small or vice versa, the path in the bifurcation diagram varies. Such a phenomenon is called hysteresis.

Example 2  Consider a one-dimensional system

\[ x' = \mu x - x^3 \quad x \in \mathbb{R} \quad (4.2) \]

where \( \mu \in \mathbb{R} \) is a bifurcating parameter.

From eq. (4.2) we know that when \( \mu \leq 0 \), eq. (4.2) has an equilibrium branch \( x = 0 \), which is asymptotically stable. When \( \mu > 0 \), eq. (4.2) has three equilibrium points, among which the point \( x = 0 \) is unstable, and the point \( x = \pm \sqrt{\mu} \) is asymptotically stable. Fig. 4.2 is the phase portrait of the equilibrium points of system (4.2), when \( \mu \) is fixed. In addition, Fig. 4.2 also shows the bifurcation diagram—the change in position of the equilibrium points of system (4.2) with change of \( \mu \). The solid line in the figure denotes the stable points (described as S), while the broken line