Chapter 4 Standard Methods for Standard Options

We now enter the part of the book that is devoted to the numerical solution of equations of the Black–Scholes type. In this chapter, we discuss “standard” options in the sense as introduced in Section 1.1 and assume the scenario characterized by the Assumptions 1.2. In case of European vanilla options the value function \( V(S, t) \) solves the Black–Scholes equation (1.2). It is not really our aim to solve this partial differential equation for vanilla payoff because it possesses an analytic solution (→ Appendix A4). Ultimately our intention is to solve more general equations and inequalities. In particular, American options will be calculated numerically. But also European options without vanilla payoff are of interest; we encounter them for Bermudan options in Section 1.8.4, and for Asian options in Section 6.3.4. The goal is not only to calculate single values \( V(S_0, 0) \) — for this purpose tree methods can be applied — but also to approximate the curve \( V(S, 0) \), or even the surface defined by \( V(S, t) \) on the half strip \( S > 0, 0 \leq t \leq T \). Thereby we collect information on early exercise, and on delta hedging by observing the derivative \( \frac{\partial V}{\partial S} \).

American options obey inequalities of the type of the Black–Scholes equation (1.2). To allow for early exercise, the Assumptions 1.2 must be weakened. As a further generalization, the payment of dividends must be taken into account; otherwise early exercise does not make sense for American calls.

The main part of this chapter outlines a PDE approach based on finite differences. We begin with unrealistically simplified boundary conditions in order to keep the explanation of the discretization schemes transparent. Later sections will discuss appropriate boundary conditions, which turn out to be tricky in the case of American options. At the end of this chapter we will be able to implement a finite-difference algorithm for standard American (and European) options. Note that this assumes constant coefficients. If we work carefully, the resulting finite-difference computer program will yield correct approximations. But the finite-difference approach is not necessarily the most efficient one. Hints on other methods will be given at the end of this chapter. For nonstandard options we refer to Chapter 6.

The classical finite-difference methods will be explained in some detail because they are the most elementary approaches to approximate differential equations. As a side-effect, this chapter serves as introduction to several fun-
damental concepts of numerical mathematics. A trained reader may like to skip Sections 4.2 and 4.3. The aim of this chapter is to introduce concepts, as well as a characterization of the free boundary (early-exercise curve), and of linear complementarity.

In addition to the finite-difference approach, “standard methods” include analytic methods, which to a significant part are based on nonnumerical analysis. The Section 4.8 will give an introduction to several such methods, including interpolation, a method of lines, and a method that solves an integral equation.

The broad field of available methods for pricing standard options calls for comparisons to judge on the relative merits of different approaches. Although such an endeavor goes beyond the scope of a text book, we offer some guidelines in Section 4.9.

4.1 Preparations

We allow for dividends paid with a continuous yield of constant level, because numerically this is a trivial extension from the case of no dividend. In case of a discrete dividend with, for example, one payment per year, a first remedy would be to convert the dividend to a continuous yield (Exercise 4.1).

A continuous flow of dividends is modeled by a decrease of \( S \) in each time interval \( dt \) by the amount \( \delta S dt \),

with a constant \( \delta \geq 0 \). This continuous dividend model can be easily built into the Black–Scholes framework. The standard model of a geometric Brownian motion represented by the SDE (1.33) is generalized to

\[
\frac{dS}{S} = (\mu - \delta) dt + \sigma dW.
\]

This is the basis for this chapter. The corresponding Black–Scholes equation for the value function \( V(S,t) \) is

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta) S \frac{\partial V}{\partial S} - r V = 0.
\] (4.1)

For constant \( r, \sigma, \delta \), this equation is equivalent to the equation

\[
\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}
\] (4.2)

\[ ^{1} \text{But the corresponding solutions } V(S,t) \text{ and their early-exercise structure will be different. The Notes and Comments summarize how to correctly compensate for a discrete dividend payment.} \]