Chapter 7 Beyond Black and Scholes

The Black–Scholes (BS) model for the value $V(S,t)$ of a vanilla option is based on some assumptions on the market. In particular, the BS model assumes the price $S_t$ of the asset on which the option is written to follow a geometric Brownian motion with a constant volatility $\sigma$. Further, transaction costs are neglected, and trading of the underlying is supposed to have no influence on the price $S_t$. As has been discussed extensively, the value function $V(S,t)$ for standard options (“plain vanilla”) of the European type, satisfies the Black–Scholes equation (1.2),

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$  \hspace{1cm} \text{(BSE)}$$

Solutions of this linear equation are subject to the terminal condition $V(S,T) = \Psi(S)$, where $\Psi$ defines the payoff.

The BS-model is the core example of a complete market. In these idealized markets, the risk exposure to variations in the underlying can be hedged away. The corresponding risk strategy is unique. Hence vanilla options modeled by Assumption 1.2 have a unique price, given by the costs of the replication strategy (→ Appendix A4). Essentially, Chapters 4 through 6 have applied numerical methods to complete markets.

For the more realistic incomplete markets, there are no perfect hedges, and a risk remains. Each hedging strategy leads to a specific model with its own price [ConT04]. The hedger compensates the remaining risk in incomplete markets by charging an additional risk premium. Hence the value function or expected value is not the price for which the option is sold. Depending on the way how the comfortable assumption of completeness of the BS-market is lost, different models are set up, calling for different numerical approaches. This Chapter 7 is devoted to computational tools for incomplete markets.

Relaxing several of the assumptions of the Black–Scholes market, nonlinear extensions of the BS equation can be derived. These “nonlinear Black–Scholes type equations” are of the form

$$\frac{\partial V}{\partial t} + \frac{1}{2} \left[ \hat{\sigma}(S,t, \frac{\partial^2 V}{\partial S^2}) \right]^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$  \hspace{1cm} (7.1)$$

In this class of models, the volatility $\hat{\sigma}$ is a function that may incorporate several types of nonlinearity. The standard PDE (BSE) is included for $\hat{\sigma} \equiv \sigma$. 


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In Section 7.1 we describe three scenarios leading to three different functions $\hat{\sigma}$ of the volatility. A nonlinear PDE as (7.1) requires special numerical treatment, which will be the focus of Section 7.2.

Another stream of research beyond Black and Scholes is devoted to jump processes (Section 7.3). One of the numerical approaches is based on partial integro-differential equations (PIDE). Some highly efficient methods apply the Fourier transform; a basic approach is discussed in Section 7.4.

### 7.1 Nonlinearities in Models for Financial Options

In this section we briefly discuss three sources of nonlinearity in (7.1). We start with transaction costs based on Leland’s approach [Lel85], and touch the more sophisticated model of Barles and Soner [BaS98]. Then we turn to specifying ranges of volatility. Finally we address the feedback by market illiquidity.

#### 7.1.1 Leland’s Model of Transaction Costs

Basic for the Black–Scholes model is the idea of rebalancing the portfolio continuously. But in financial reality this continuous trading would cause arbitrarily high trading costs. Keeping transaction costs low forces to abandon the optimal Black–Scholes hedging. But without the ideal BS hedging, the model suffers from hedging errors. To compromise, the hedger searches a balance between keeping both the transaction costs low and the hedging errors low.

Suppose that instead of rebalancing continuously, trading is only possible at discrete time instances with time step $\Delta t$ apart ($\Delta t$ fixed and finite). We assume a transaction cost rate proportional to the trading volume $\nu S$:

$$\text{trading } \nu \text{ assets costs the amount } c|\nu|S$$

for some cost parameter $c$.

Here we sketch a heuristic derivation of a model due to [Lel85], [HoWW94]. The discussion of this model parallels that for the Black–Scholes model, now adapted to the discrete scenario.\(^1\) The stochastic changes of the asset with price $S$ and of a riskless bond with price $B$ are

$$\Delta S = \mu S \Delta t + \sigma S \Delta W$$
$$\Delta B = r B \Delta t.$$

The portfolio with value $\Pi$ is taken in the form

\(^1\) All other BS-assumptions remain untouched [Kwok98]. The following analysis uses or modifies Appendix A4 with (A4.1), (A4.3), (A4.8). $\Delta$ means the increment, and not the greek $\frac{\partial V}{\partial S}$. 