As our goal is to study the geometry of the hyperbolic plane by considering quantities invariant under the action of a reasonable group of transformations, we spend this chapter by describing such a reasonable group of transformations of \( \mathbb{C} \), namely the \textit{general Möbius group} \( \text{Möb} \), which consists of compositions of \textit{Möbius transformations} and \textit{reflections}. We close the chapter by restricting our attention to the transformations in \( \text{Möb} \) preserving \( \mathbb{H} \).

\section*{2.1 The Group of Möbius Transformations}

Since every hyperbolic line in \( \mathbb{H} \) is by definition contained in a circle in \( \mathbb{C} \), we begin the process of determining the transformations of \( \mathbb{H} \) taking hyperbolic lines to hyperbolic lines by first determining the group of homeomorphisms of \( \mathbb{C} \) taking circles in \( \mathbb{C} \) to circles in \( \mathbb{C} \).

For the sake of notational convenience, let \( \text{Homeo}^C(\mathbb{C}) \) be the subset of the group \( \text{Homeo}(\mathbb{C}) \) of homeomorphisms of \( \mathbb{C} \) that contains all those homeomorphisms of \( \mathbb{C} \) taking circles in \( \mathbb{C} \) to circles in \( \mathbb{C} \).

Note that, while it is easy to see that the composition of two elements of \( \text{Homeo}^C(\mathbb{C}) \) is again an element of \( \text{Homeo}^C(\mathbb{C}) \) and that the identity homeomorphism is an element of \( \text{Homeo}^C(\mathbb{C}) \), we do not yet know that inverses of
elements of Homeo$^C(\overline{\mathbb{C}})$ lie in Homeo$^C(\overline{\mathbb{C}})$, and hence we cannot yet conclude that Homeo$^C(\overline{\mathbb{C}})$ is a group.

In fact, there are many homeomorphisms of $\overline{\mathbb{C}}$ that do not lie in Homeo$^C(\overline{\mathbb{C}})$.

**Exercise 2.1**

Give an explicit example of an element of Homeo$(\overline{\mathbb{C}})$ that is not an element of Homeo$^C(\overline{\mathbb{C}})$.

We begin by considering a class of homeomorphisms of $\overline{\mathbb{C}}$ that we understand, namely those arising from polynomials. As we saw in Exercise 1.14 and Exercise 1.15, to each polynomial $g(z)$ we may associate the function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ given by

$$f(z) = g(z) \text{ for } z \in \mathbb{C} \text{ and } f(\infty) = \infty.$$  

As we wish to consider homeomorphisms of $\overline{\mathbb{C}}$ which arise from polynomials, we restrict our attention to polynomials of degree 1.

**Proposition 2.1**

The element $f$ of Homeo$(\overline{\mathbb{C}})$ defined by

$$f(z) = az + b \text{ for } z \in \mathbb{C} \text{ and } f(\infty) = \infty,$$

where $a, b \in \mathbb{C}$ and $a \neq 0$, is an element of Homeo$^C(\overline{\mathbb{C}})$.

Recall from Section 1.2 that each circle $A$ in $\overline{\mathbb{C}}$ is the set of solutions to an equation of the form

$$\alpha z \overline{z} + \beta z + \overline{\beta \overline{z}} + \gamma = 0,$$

where $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$, and where $\alpha \neq 0$ if and only if $A$ is a circle in $\mathbb{C}$.

We begin with the case that $A$ is a Euclidean line in $\mathbb{C}$. So, consider the Euclidean line $A$ given as the solution to the equation

$$A = \{ z \in \mathbb{C} \mid \beta z + \overline{\beta \overline{z}} + \gamma = 0 \},$$

where $\beta \in \mathbb{C}$ and $\gamma \in \mathbb{R}$. We wish to show that if $z$ satisfies this equation, then $w = az + b$ satisfies a similar equation.

Since $w = az + b$, we have that $z = \frac{1}{a}(w - b)$. Substituting this into the equation for $A$ given above gives

$$\beta z + \overline{\beta \overline{z}} + \gamma = \beta \frac{1}{a}(w - b) + \overline{\beta \frac{1}{a}(w - b)} + \gamma$$

$$= \frac{\beta}{a}w + \left(\frac{\beta}{a}\right)\overline{w} - \beta \frac{b}{a} - \overline{\beta}b + \gamma = 0.$$