As our goal is to study the geometry of the hyperbolic plane by considering quantities invariant under the action of a reasonable group of transformations, we spend this chapter by describing such a reasonable group of transformations of $\mathbb{C}$, namely the general Möbius group $\text{Möb}$, which consists of compositions of $\text{Möbius transformations}$ and $\text{reflections}$. We close the chapter by restricting our attention to the transformations in $\text{Möb}$ preserving $\mathbb{H}$.

### 2.1 The Group of Möbius Transformations

Since every hyperbolic line in $\mathbb{H}$ is by definition contained in a circle in $\mathbb{C}$, we begin the process of determining the transformations of $\mathbb{H}$ taking hyperbolic lines to hyperbolic lines by first determining the group of homeomorphisms of $\mathbb{C}$ taking circles in $\mathbb{C}$ to circles in $\mathbb{C}$.

For the sake of notational convenience, let $\text{Homeo}^C(\mathbb{C})$ be the subset of the group $\text{Homeo}(\mathbb{C})$ of homeomorphisms of $\mathbb{C}$ that contains all those homeomorphisms of $\mathbb{C}$ taking circles in $\mathbb{C}$ to circles in $\mathbb{C}$.

Note that, while it is easy to see that the composition of two elements of $\text{Homeo}^C(\mathbb{C})$ is again an element of $\text{Homeo}^C(\mathbb{C})$ and that the identity homeomorphism is an element of $\text{Homeo}^C(\mathbb{C})$, we do not yet know that inverses of...
elements of $\text{Homeo}^C(\overline{\mathbb{C}})$ lie in $\text{Homeo}^C(\overline{\mathbb{C}})$, and hence we cannot yet conclude that $\text{Homeo}^C(\overline{\mathbb{C}})$ is a group.

In fact, there are many homeomorphisms of $\overline{\mathbb{C}}$ that do not lie in $\text{Homeo}^C(\overline{\mathbb{C}})$.

**Exercise 2.1**

Give an explicit example of an element of $\text{Homeo}(\overline{\mathbb{C}})$ that is not an element of $\text{Homeo}^C(\overline{\mathbb{C}})$.

We begin by considering a class of homeomorphisms of $\overline{\mathbb{C}}$ that we understand, namely those arising from polynomials. As we saw in Exercise 1.14 and Exercise 1.15, to each polynomial $g(z)$ we may associate the function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ given by

$$f(z) = g(z) \text{ for } z \in \mathbb{C} \text{ and } f(\infty) = \infty.$$ 

As we wish to consider homeomorphisms of $\overline{\mathbb{C}}$ which arise from polynomials, we restrict our attention to polynomials of degree 1.

**Proposition 2.1**

The element $f$ of $\text{Homeo}(\overline{\mathbb{C}})$ defined by

$$f(z) = az + b \text{ for } z \in \mathbb{C} \text{ and } f(\infty) = \infty,$$

where $a, b \in \mathbb{C}$ and $a \neq 0$, is an element of $\text{Homeo}^C(\overline{\mathbb{C}})$.

Recall from Section 1.2 that each circle $A$ in $\overline{\mathbb{C}}$ is the set of solutions to an equation of the form

$$az \overline{z} + \beta z + \overline{\beta \overline{z} + \gamma = 0},$$

where $a, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$, and where $a \neq 0$ if and only if $A$ is a circle in $\mathbb{C}$.

We begin with the case that $A$ is a Euclidean line in $\mathbb{C}$. So, consider the Euclidean line $A$ given as the solution to the equation

$$A = \{z \in \mathbb{C} | \beta z + \overline{\beta \overline{z} + \gamma = 0}\},$$

where $\beta \in \mathbb{C}$ and $\gamma \in \mathbb{R}$. We wish to show that if $z$ satisfies this equation, then $w = az + b$ satisfies a similar equation.

Since $w = az + b$, we have that $z = \frac{1}{a}(w - b)$. Substituting this into the equation for $A$ given above gives

$$\beta z + \overline{\beta \overline{z} + \gamma = \beta \frac{1}{a}(w - b) + \overline{\beta \frac{1}{a}(w - b)} + \gamma = \frac{\beta}{a}w + \frac{\overline{\beta}}{a} \overline{w - \frac{\beta}{a}b - \frac{\overline{\beta}}{a}b + \gamma = 0}.$$