A fundamental result for linear continuous maps is the uniform boundedness principle. It states that the pointwise boundedness of a family of operators already implies their boundedness in the operator norm. This principle rests upon the following theorem.

### 7.1 Baire’s category theorem

Let $X$ be a nonempty complete metric space and let

$$X = \bigcup_{k \in \mathbb{N}} A_k,$$

with closed sets $A_k \subset X$.

Then there exists a $k_0 \in \mathbb{N}$ with $\bar{A}_{k_0} \neq \emptyset$.

**Remark:** Recall that $\bar{A}_k = \text{intr}_X(A_k)$.

**Proof.** Assume that $\bar{A}_k = \emptyset$ for all $k$. Then

$U \subset X$ open, nonempty, $k \in \mathbb{N}$

$\implies$ $U \setminus A_k$ open, nonempty

$\implies$ there exists a ball $B_\varepsilon(x) \subset U \setminus A_k$ with $\varepsilon \leq \frac{1}{k}$.

Hence we can inductively choose balls $B_{\varepsilon_k}(x_k)$ such that

$$\bar{B}_{\varepsilon_k}(x_k) \subset B_{\varepsilon_{k-1}}(x_{k-1}) \setminus A_k \quad \text{and} \quad \varepsilon_k \leq \frac{1}{k}.$$

Consequently, we see that $x_l \in B_{\varepsilon_k}(x_k)$ for $l \geq k$ and $\varepsilon_k \to 0$ as $k \to \infty$ and the balls $B_{\varepsilon_k}(x_k)$ are nested, and we conclude that $(x_l)_{l \in \mathbb{N}}$ is a Cauchy sequence. Hence there exists the limit

$$x := \lim_{l \to \infty} x_l \in X$$

and $x \in \bar{B}_{\varepsilon_k}(x_k)$ for all $k$. Since $\bar{B}_{\varepsilon_k}(x_k) \cap A_k = \emptyset$, we have that

$$x \notin \bigcup_{k \in \mathbb{N}} A_k = X,$$

a contradiction. \qed
As is evident from the example $X = \mathbb{Q}$, the completeness assumption in 7.1 is essential. With the help of 7.1 we can now show the following:

### 7.2 Theorem (Uniform boundedness principle).

Let $X$ be a nonempty complete metric space and let $Y$ be a normed space. Let $\mathcal{F} \subset C^0(X; Y)$ be a set of functions with

$$\sup_{f \in \mathcal{F}} \| f(x) \|_Y < \infty \quad \text{for every } x \in X.$$  

(7-5)

Then there exist an $x_0 \in X$ and an $\varepsilon_0 > 0$, such that

$$\sup_{x \in B_{\varepsilon_0}(x_0)} \sup_{f \in \mathcal{F}} \| f(x) \|_Y < \infty.$$  

(7-6)

**Proof.** For $f \in \mathcal{F}$ and $k \in \mathbb{N}$ it holds that $\{ x \in X ; \| f(x) \|_Y \leq k \}$ is a closed set. Hence the sets

$$A_k := \bigcap_{f \in \mathcal{F}} \{ x \in X ; \| f(x) \|_Y \leq k \}$$

being intersections of closed sets, are closed, and it follows from (7-5) that they form a cover of $X$. Then theorem 7.1 yields that $\hat{A}_{k_0} \neq \emptyset$ for some $k_0$, and hence there exists a $\overline{B}_{\varepsilon_0}(x_0) \subset A_{k_0}$. Noting that

$$\sup_{x \in A_{k_0}} \sup_{f \in \mathcal{F}} \| f(x) \|_Y \leq k_0$$

yields the desired result. \qed

For linear continuous maps 7.2 is reformulated as the

### 7.3 Banach-Steinhaus theorem.

Let $X$ be a Banach space and let $Y$ be a normed space. Suppose $\mathcal{T} \subset \mathcal{L}(X; Y)$ with

$$\sup_{T \in \mathcal{T}} \| Tx \|_Y < \infty \quad \text{for every } x \in X.$$  

Then $\mathcal{T}$ is a bounded set in $\mathcal{L}(X; Y)$, i.e.

$$\sup_{T \in \mathcal{T}} \| T \|_{\mathcal{L}(X; Y)} < \infty.$$  

**Proof.** Setting

$$f_T(x) := \| Tx \|_Y \quad \text{for } T \in \mathcal{T}, \ x \in X$$

defines functions $f_T \in C^0(X; \mathbb{R})$, and $\mathcal{F} := \{ f_T ; T \in \mathcal{T} \}$ satisfies the assumptions in 7.2. Hence, by the conclusions of 7.2, there exist an $x_0 \in X$, an $\varepsilon_0 > 0$, and a constant $C < \infty$ with