9 Finite-dimensional approximation

In this chapter we consider certain finite-dimensional subspaces of Banach spaces. This plays an important role in applications, where we regard elements in such subspaces as approximations of elements in the entire Banach space $X$. Clearly we require the approximating subspaces to be finite-dimensional, because in numerical computations only a prescribed finite number of coordinates can be stored.

Hence in what follows points $x \in X$ will always be approximated by a countable sequence of points $(x_n)_{n \in \mathbb{N}}$. It is for this reason that in applications the weak and weak* sequential compactness introduced in Chapter 8 plays a far more important role than compactness with respect to the weak and weak* topology, respectively.

An important optimization problem is to characterize a function from a function space $X$ approximately by finitely many numerical values. This can be achieved by suitably exhausting $X$ by finite-dimensional subspaces $X_n$, $n \in \mathbb{N}$. Another important problem is to numerically solve linear equations between Banach spaces. This concerns for instance the numerical solution of a boundary value problem for linear partial differential equations (see 6.5). Here it is once again necessary to approximate a Banach space, e.g. the space $X = W^{1,2}(\Omega)$, by suitable finite-dimensional subspaces $X_n$, $n \in \mathbb{N}$, and then to find an approximative solution in these subspaces (see the Ritz-Galerkin method in 9.23–9.25). In all applications the subspaces $X_n$ are chosen so that they are appropriate for the problem at hand, i.e. they should be easy to handle and, on the other hand, they should also retain as much of the structure of the infinite-dimensional problem as possible.

To approximate a space by countably many finite-dimensional subspaces is only possible for separable normed spaces:

9.1 Lemma. Let $X$ be an infinite-dimensional normed space. Then the following are equivalent:

1. $X$ is separable.
2. There exist finite-dimensional subspaces $X_n \subset X$ for $n \in \mathbb{N}$ such that $X_n \subset X_{n+1}$ for all $n$ and

$$\bigcup_{n \in \mathbb{N}} X_n$$

is dense in $X$. 

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(3) There exist finite-dimensional subspaces $E_n \subset X$ for $n \in \mathbb{N}$ such that $E_n \cap E_m = \{0\}$ for $n \neq m$ and
\[
\bigoplus_{k \in \mathbb{N}} E_k := \bigcup_{n \in \mathbb{N}} \left( E_1 \oplus \cdots \oplus E_n \right) \text{ is dense in } X.
\]

(4) There exists a linearly independent set $\{e_k; k \in \mathbb{N}\} \subset X$ such that $\text{span}\{e_k; k \in \mathbb{N}\}$ is dense in $X$.

**Proof (1)⇒(2).** Choose a countable set $\{x_n; n \in \mathbb{N}\}$ that is dense in $X$. Define $X_n := \text{span}\{x_1, \ldots, x_n\}$. □

**Proof (2)⇒(3).** Let $E_1 := X_1$ and for $n \in \mathbb{N}$ choose subspaces $E_{n+1}$ with $X_{n+1} = X_n \oplus E_{n+1}$. □

**Proof (3)⇒(4).** Let $d_n := \dim E_n$ and let $\{e_{n,j}; j = 1, \ldots, d_n\}$ be a basis of $E_n$. Then we have that
\[
X_n := E_1 \oplus \cdots \oplus E_n = \text{span}\{e_{i,j}; 1 \leq i \leq n, 1 \leq j \leq d_i\}
\]
and hence $M := \{e_{i,j}; i \in \mathbb{N}, 1 \leq j \leq d_i\}$ is a desired linearly independent set, since
\[
\text{span}(M) = \text{span}\left( \bigcup_{n \in \mathbb{N}} X_n \right)
\]
is a dense set in $X$. □

**Proof (4)⇒(1).** For $n \in \mathbb{N}$ it holds that
\[
A_n := \left\{ \sum_{k=1}^{n} \alpha_k e_k; \alpha_k \in Q \text{ for } 1 \leq k \leq n \right\},
\]
with $Q$ as in the proof of 4.17(4), is countable with $\text{clos}(A_n) = \text{span}\{e_k; 1 \leq k \leq n\}$. The desired result then follows from 4.17(1). □

On recalling 4.18(4) we can apply these results, for example, to the spaces $L^p(\Omega)$, $1 \leq p < \infty$, with $\Omega \subset \mathbb{R}^n$ open. Moreover, by 4.15(3), we have $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$, and so 4.17(2) yields that it is also separable with respect to the $L^p$-norm. Hence we can apply 9.1 to the space $C_0^\infty(\Omega)$ equipped with the $L^p$-norm. The denseness results in 9.1(2)-9.1(4) for this space then also hold with respect to the space $L^p(\Omega)$. This implies that the finite-dimensional spaces with respect to $L^p(\Omega)$ in 9.1 can be chosen as subspaces of $C_0^\infty(\Omega)$. The same argument can equally be applied to other pairs of function spaces.

As a further consequence of 9.1 we now give a constructive proof of the Hahn-Banach theorem 6.15 for separable spaces $X$, i.e. for the extension of a functional $y' \in Y'$ onto $X$ for a subspace $Y \subset X$. Recalling that the extension of $y'$ to the closure $\overline{Y}$ is already uniquely defined by $y'$ (see E5.3), we restrict our attention to closed subspaces $Y \subset X$. 