7 Optimal Portfolio Rules

Symbolic Solutions of Stochastic Control Problems in Portfolio Management

7.1 Remarks

The classical portfolio theory goes back to Markowitz and his mean-variance portfolio theory. Portfolio theory based on stochastic control goes back to Merton's classical paper in the early 70s [see, e.g., Ch. 5 of 46].

This chapter is an overview, Mathematica® implementation, and an extension of the Merton theory. All the results are based on the Log-Normal asset-price dynamics. This assumption yields beautiful explicit/symbolic solutions that are extremely efficient computationally. On the other hand, the same assumption is a source of criticism, since the asset-price dynamics Log-Normality is often not sufficiently descriptive. We shall remove this assumption completely in the next Chapter 8 (Section 8.2), where we introduce a new theory for optimal portfolio hedging for assets with general SDE price dynamics based on numerical solutions of reduced Monge–Ampère PDEs.

In our search for explicit/symbolic solutions to portfolio optimization problems we shall employ two methods. One is by finding explicit solutions of Monge–Ampère PDEs, while the other way is by direct maximization of the expected gain with respect to the portfolio rules. The first method is based on partial differential equations, while the second is probability theory based, although it is facilitated by some ordinary differential equations, and even by calculus of variations. One method will be checking the other, and/or continuing when the other cannot be applied.

Explicit/symbolic solutions of portfolio optimization problems, in addition to being very fast computationally and therefore potentially very convenient for day to day trading computations, have high educational value. By understanding and solving them in detail, and experimenting, one can grasp what the right stochastic control problems are to pose also in more complicated situations when each experiment is expensive computationally. One can also better determine reasonable parameter values, the kind of constraints needed and what to expect from them. Finally, one
comes to understand the interplay among different kinds of constraints, the connection between utility functions and various constraints, and so on.

### 7.2 Utility of Wealth

At first it may appear quite surprising how much of what follows depends on the notion of (and on the selection of) the utility function. The utility function measures the utility of wealth. We shall always denote the total wealth or the total balance in the investor's brokerage account at time $t$ by $X = X(t)$. Now one may notice right away that what $X(t)$ is exactly makes a whole lot of difference. It is quite different from the point of view of how much risk it is prudent to take: whether $X(t)$ is a gambling allowance, the balance in one of say several brokerage accounts that an investor holds, or if the risk involves the total family fortune. Alternatively, as we shall see in detail, explicit mathematical constraints on the investment portfolio may change the nature of $X(t)$ and the amount of risk one can afford to take.

So what is the proper way of measuring the utility of wealth, i.e., risk-taking? In a way, the entire chapter that follows is, in addition to many other issues, also about answering this question posteriori, meaning that after calculating the strategy implied by a particular utility function, we can judge whether it seems reasonable (see for example final thoughts in Section 7.4.1, and even Section 8.2), and whether the experimental wealth evolutions it generates seem acceptable, thereby judging the utility function used.

On the other hand, a priori, from the psychological point of view, or alternatively, from the trading experience, it is known that the pain experienced by the loss of $-dX$ is greater than the pleasure experienced from the gain of $dX > 0$. This is particularly true if $X$ is the total wealth and no additional constraints on risk-taking are imposed. This fact translates into the mathematical property of (strict) concavity for any reasonable utility function under such circumstances. If, on the other hand, for example, $X$ is the balance in an account among several accounts and is designated for risk-taking or experimenting, then a different much more aggressive utility function may be appropriate.

There exists a body of literature that considers, in addition to accumulation of the wealth also its consumption, both in an optimal manner balancing the two. Although we could, using the same methods that are presented in this chapter and in Section 8.2, we do not consider the issue of wealth consumption, but only the issue of wealth accumulation. Consequently, the most important class of utility functions to be considered here is the class of functions:

\[
\ln[1]:= \psi_{\gamma}[X_\gamma] := \frac{X^{1-\gamma}}{1-\gamma}
\]

for $0 < \gamma \neq 1$, while for $\gamma = 1$

\[
\ln[2]:= \psi_1[X_\gamma] := \log[X]
\]

Notice right away that

\[
\ln[3]:= \partial_{(X,2)} \psi_{\gamma}[X]
\]