13
Newton’s Method for Riccati Equations

13.1 Newton’s Method

Newton’s method is an iterative procedure for solving nonlinear (systems of) equations. It is based on replacing a nonlinear operation by its linearization, and then updating the approximation to the function zero by solving the linear equations that result.

For a single nonlinear equation of the form

\[ f(x) = 0 \]

the algorithm basis is the approximation

\[ f(x_n + (x_{n+1} - x_n)) \approx f(x_n) + f'(x_n)(x_{n+1} - x_n). \]

Replacing the approximate left side by the desired value of 0 gives

\[ 0 = f(x_n) + f'(x_n)(x_{n+1} - x_n), \]
\[ x_{n+1} = x_n - \frac{1}{f'(x_n)} f(x_n). \]

The same idea is applicable to a system of nonlinear equations represented in the form

\[ F(x) = 0. \]

The linear approximation is

\[ F(x_n + (x_{n+1} - x_n)) \approx F(x_n) + \frac{\partial F}{\partial x}(x_n)(x_{n+1} - x_n), \]
where $\frac{\partial F}{\partial x}$ represents the Jacobian matrix of the nonlinear transformation. The natural problem restriction is that this is invertible, leading to the multivariable iteration

$$x_{n+1} = x_n - \frac{\partial F}{\partial x}(x_n)^{-1}F(x_n).$$

The multivariable version uses a notation that is strongly biased by the idea that the unknowns are represented in the form of a (column) vector. The interest in this chapter is in treating the steady state Riccati equations

$$0 = A^* K + KA - KBR^{-1}B^*K + L$$

and

$$K = A^*KA - A^*KB[R + B^*KB]^{-1}B^*KA^* + L$$

by means of Newton's method. In this situation, the unknown $K$ is matrix-valued, and although the unknown components could be structured into a column of unknowns, it is more convenient to construct a Newton iteration in the "original" matrix format. We use the notation $L$ for the "constant term" of the Riccati equation to slightly abbreviate the notation.

### 13.2 Continuous Time Riccati Equations

The continuous time steady state Riccati equation can be represented as

$$F(K) = 0,$$

where $F$ is defined by (using $L = C^*QC$ for economy)

$$F(K) = A^* K + KA - KBR^{-1}B^*K + L.$$  

Compute

$$F(K + \Delta K) = A^* (K + \Delta K) + (K + \Delta K)A$$

$$-(K + \Delta K)BR^{-1}B^*(K + \Delta K) + L.$$  

Then

$$F(K + \Delta K) - F(K) = A^* \Delta K + \Delta KA - \Delta KBR^{-1}B^*K - KBR^{-1}B^*\Delta K + O(\Delta K^2).$$

$$= \{A^* - KBR^{-1}B^*\} \Delta K + \Delta K\{A - BR^{-1}B^*K\} + O(\Delta K^2).$$

The above expression can be regarded as providing the linearization of $F$, but expressed in "natural" matrix terms rather than as a Jacobian. Symbolically, the linear approximation mapping is defined by

$$F'(K) \circ \Delta K = \{A^* - KBR^{-1}B^*\} \Delta K + \Delta K\{A - BR^{-1}B^*K\}.$$